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A discrete and *q* asymptotic iteration method

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ABSTRACT

We introduce a finite difference and q-difference analogues of the Asymptotic Iteration Method of Ciftci, Hall, and Saad. We give necessary, and sufficient condition for the existence of a polynomial solution to a general linear second-order difference or q-difference equation subject to a 'terminating condition', which is precisely defined. When a difference or q-difference equation has a polynomial solution, we show how to find the second solution.

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1. Introduction

The problem of finding polynomial solutions to differential equations of Sturm–Liouville type goes back to the 19th century. Routh [23] essentially solved the problem of finding orthogonal polynomial solutions to differential equations of the form

$$f(x)y''(x) + g(x)y'(x) + h(x)y(x) = \lambda_n y(x) \tag{1}$$

where f, g and h are polynomials and λ_n is the spectral parameter. He demanded that the (1) has polynomial solutions of degree n for $n = 0, 1, \ldots, N$, where N is a fixed number > 1, or is $+\infty$. Earlier Heine [25, Section 6.8] considered polynomial solutions to a differential equation of the form

$$f(x)y''(x) + g(x)y'(x) + h(x)y(x) = 0. \tag{2}$$

where f, and g are given polynomials of degrees at most $p+1$ and p, respectively, while h is a polynomial of degree $p-1$, to be determined in order for Equation (2) to have a polynomial solution of a prescribed degree n. Stieltjes, motivated by an electrostatic equilibrium problem [12, Chapter 3], also studied this problem. The polynomials h in (2) are called Van Vleck polynomials and the polynomial solution to (2) is called a Stieltjes polynomials. This theory is well-explained in Section 9 of Marden's excellent monograph [21]. When f has real and simple zeros which interlace with the zeros of g the theory further

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simplifies, see § 6.8 in [25]. In Bochner [3] characterized all polynomial solutions (not necessarily orthogonal) to (1) with $N = \infty$. Routh's theorem was extended to the difference, or q-difference operators, see the survey article [1]. A more general treatment is in Chapter 20 of [12], where the corresponding problem for the Askey–Wilson operator is also mentioned. A recent variation on the Routh (or Bochner) problem was introduced in the works [8–10] by D. Gómez-Ullate, N. Kamran, R. Milson. They looked for equations of the type (1) but they demanded them to have orthogonal polynomials solutions of degree n, for all $n \geq m$ for some m. This investigation generated what is now called exceptional orthogonal polynomials.

The Asymptotic Iteration Method (AIM) was introduced in 2003 in [5,24], see also [4], as a tool to find closed form solution to a fairly large class of second-order differential equations. The method has been applied to a variety of problems and seems to provide new insight into an old problem [27].

We felt that working out a discrete and a q-analogue of AIM is a worthwhile endeavour and this paper indeed provides a discrete and a q- analogue of AIM, which we refer to as DAIM and q-AIM. The techniques used in both cases are almost parallel, so we included a detailed treatment of DAIM but only sketched the outline of q-AIM. We give some examples to illustrate the power of this approach.

Section 2 contains a brief list of definitions and the notations used in this work. In Section 3 we introduce the discrete version of AIM, called DAIM. In it, we show how to construct two linearly independent solutions of a general linear second order difference equation with variable coefficients under the assumption (27), which we shall call a terminating condition. In Section 4 we prove that the general linear second order difference equation has a polynomial solution if and only if the terminating condition (27) holds for some n. In Section 5 we give several examples including Euler-type equations and the discrete version of the hypergeometric equation. Section 6 treats the linear second-order q-difference equations where we derive the theory qAIM in parallel with the DAIM technique. We also characterize q-difference equations which have a polynomial solution regarding a terminating condition. Section 7 we implement the q-AIM technique to explore several examples including the q-Laguerre difference equation, Al-Salam-Carlitz q-difference equation, and the Stieltjes-Wigert q-difference equation. Section 8 discusses the limitations of the AIM, DAIM, and q-AIM method.

It is worth mentioning that the Heine and Stieltjes theories for differential equations with polynomial solutions have not been extended to the difference or q-difference equations. It will be interesting to develop such a theory.

Remark 1.1: The difference or q-difference equations we consider have parameters. One important point is that it may be easy to find necessary conditions on the parameters in order for the equation to have a polynomial solution. Our approach gives necessary and sufficient conditions for a polynomial solution to exist.

2. Preliminaries for difference and q-difference equations

It easy to see that the problem

$$y(n+1) = \lambda(n)y(n) + g(n), \quad y(n_0) = y_0, \quad n \geq n_0 \geq 0. \tag{3}$$

when $\lambda(n) \neq 0$ for all n, has the solution

$$y(n) = \left(\prod_{i=n_0}^{n-1} \lambda(i)\right) y_0 + \sum_{i=n_0}^{n-1} \left[\left(\prod_{\ell=i+1}^{n-1} \lambda(\ell)\right) g(i)\right]. \tag{4}$$

We shall use the standard notation for the finite difference operators E, Δ, ∇ as in [16,22]. In general, for $n = 1, 2, \ldots,$ we have

$$\Delta^n f(x) = ((E - I)^n f)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + n - k), \tag{5}$$

$$\nabla^n f(x) = ((I - E^{-1})^n f)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x - k). \tag{6}$$

Some of the formulas used in the sequel are:

$$\nabla^k f(x + k) = \Delta^k f(x), \quad \Delta^k f(x - k) = \nabla^k f(x), \quad k = 1, 2, \ldots, \tag{7}$$

$$\Delta\nabla f(x) = \nabla\Delta f(x) = f(x + 1) - 2f(x) + f(x - 1) = (\Delta - \nabla) f(x). \tag{8}$$

The product rule is

$$\Delta[f(x)g(x)] = g(x)\Delta f(x) + f(x + 1)\Delta g(x) \tag{9}$$

$$= f(x)\Delta g(x) + g(x)\Delta f(x) + \Delta f(x)\Delta g(x). \tag{10}$$

The quotient rule is

$$\Delta\left(\frac{g(x)}{f(x)}\right) = \frac{f(x)\Delta g(x) - g(x)\Delta f(x)}{f(x)f(x + 1)}. \tag{11}$$

The symmetric Leibniz rule for finite difference operators is [22]

$$(\Delta^n fg)(x) = n! \sum_{j,k \geq 0,\, j+k \leq n} \frac{(\Delta^j f)(x)(\Delta^k g)(x)}{j!k!(n - j - k)!}. \tag{12}$$

The notation for q-shifted factorials is [2,7]

$$(a; q)_0 := 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - a q^j), \quad n = 1, 2, \ldots, \text{ or } \infty. \tag{13}$$

Here we always assume that $0 < q < 1$. The q-analogue of the binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \tag{14}$$

We also have

$$(1 - q)^n x^n (D_q^n f)(x) = q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} f(xq^{n-k}). \tag{15}$$

The product and quotient rules are

$$D_q[f(x)g(x)] = g(x)D_qf(x) + f(qx)D_qg(x), \tag{16}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(qx)g(x)}. \tag{17}$$

Let $\alpha(x)$ be continuous at $x = 0$. Then the solution to

$$(D_qy)(x) = \alpha(x)y(x), \tag{18}$$

which is continuous at $x = 0$ is

$$y(x) = \frac{y(0)}{\prod_{k=0}^{\infty}[1-(1-q)x\,q^k\alpha(xq^k)]}. \tag{19}$$

This follows trivially. Moreover if $\alpha(x)$ and $\beta(x)$ are continuous at $x = 0$ then the solution to

$$(D_qy)(x) = \alpha(x)y(x) + \beta(x), \tag{20}$$

which is continuous at $x = 0$, is given by

$$y(x) = \frac{y(0)}{\prod_{k=0}^{\infty}[1-(1-q)xq^k\alpha(xq^k)]} + \sum_{k=0}^{\infty}\frac{xq^k(1-q)\beta(xq^k)}{\prod_{j=0}^{k}[1-(1-q)xq^j\alpha(xq^j)]}. \tag{21}$$

If $y(x)$ satisfies a linear homogeneous difference equation then $f(x)y(x)$ will satisfy the same equation if f is unit periodic, that is $f(x+1) = f(x)$. Thus unit periodic functions play the role played by constants in the theory of differential equations. Similarly functions satisfying $f(qx) = f(x)$ play the role of constants in the theory of q-difference equations.

3. Discrete asymptotic iteration method (DAIM)

The second-order difference equation may take one of the following forms

$$\Delta^2y(x) = \lambda_0(x)\Delta y(x) + s_0(x)y(x), \tag{22}$$

$$\Delta\nabla y(x) = \alpha_0(x)\Delta y(x) + \beta_0(x)y(x), \tag{23}$$

$$\nabla\Delta y(x) = \alpha_1(x)\nabla y(x) + \beta_1(x)y(x), \tag{24}$$

These forms are equivalent and we shall focus our attention on the first form (22).

Unlike the original form of AIM where the boundary conditions contributed in setting up the asymptotic solution, in the discrete version the initial conditions must be incorporated within the development of the analytic solution at later stage.

Theorem 3.1: *If $y(x)$ satisfies* (22), *then*

$$\Delta^{n+2}y(x) = \lambda_n(x)\Delta y(x) + s_n(x)y(x), \tag{25}$$

where

$$\begin{aligned} \lambda_n(x) &= \Delta\lambda_{n-1}(x) + \lambda_{n-1}(x+1)\lambda_0(x) + s_{n-1}(x+1), \quad n > 0, \\ s_n(x) &= \Delta s_{n-1}(x) + \lambda_{n-1}(x+1)s_0(x), \quad n > 0. \end{aligned} \tag{26}$$

Proof: The proof is by induction on n. ∎

We note that the above mentioned construction is reminiscent of the construction of the Lommel polynomials from the three-term recurrence relation of the Bessel functions given in Watson [26] and is reproduced in [12]. The q-Lommel polynomials associated with $J_\nu^{(2)}$ was given in [11] while the construction associated with $J_\nu^{(3)}$ was given in [18].

Theorem 3.2: *Let λ_n and s_n be as in* (26), *and set* $\delta_n(x) = \lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)\, s_n(x)$. *If* $\delta_n(x) = 0$, *then* $\delta_m(x) = 0$ *for all* $m \geq n$.

Proof: It suffices to show that if $\delta_n(x) = 0$, then $\delta_{n+1}(x) = 0$. Using the definition (26), we find that

$$\begin{aligned}
\delta_{n+1}(x) &= \lambda_{n+1}(x)s_n(x) - \lambda_n(x)s_{n+1}(x) \\
&= s_n(x)\Delta\lambda_n(x) - \lambda_n(x)\Delta s_n(x) + \lambda_n(x+1)s_n(x)\lambda_0(x) + s_n(x+1)s_n(x) \\
&\quad - \lambda_n(x)\lambda_n(x+1)s_0(x) \\
&= \left(\frac{s_n(x)\Delta\lambda_n(x) - \lambda_n(x)\Delta s_n(x)}{s_n(x)s_n(x+1)}\right) s_n(x)s_n(x+1) + \lambda_n(x+1)(s_n(x)\lambda_0(x) \\
&\quad - \lambda_0(x)s_0(x)) + s_n(x+1)s_n(x) \\
&= \Delta\left(\frac{\lambda_n(x)}{s_n(x)}\right) s_n(x)s_n(x+1) + \lambda_n(x+1)s_n(x)\lambda_0(x) + s_n(x+1)s_n(x) \\
&\quad - \lambda_n(x)\lambda_n(x+1)s_0(x) \\
&= s_n(x)s_n(x+1)\left(\Delta\left(\frac{\lambda_n(x)}{s_n(x)}\right) + 1 + \frac{\lambda_n(x+1)}{s_n(x+1)}\lambda_0(x) - \frac{\lambda_n(x)\lambda_n(x+1)}{s_n(x)s_n(x+1)}s_0(x)\right) \\
&= s_n(x)s_n(x+1)\left(\Delta\left(\frac{\lambda_{n-1}(x)}{s_{n-1}(x)}\right) + 1\right. \\
&\quad \left. + \frac{\lambda_{n-1}(x+1)}{s_{n-1}(x+1)}\left(\lambda_0(x) - \frac{\lambda_{n-1}(x)}{s_{n-1}(x)}s_0(x)\right)\right) \\
&= s_n(x)s_n(x+1)\left(\frac{s_{n-1}(x)\Delta\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta s_{n-1}(x)}{s_{n-1}(x)s_{n-1}(x+1)} + 1\right. \\
&\quad \left. + \frac{\lambda_{n-1}(x+1)}{s_{n-1}(x+1)}\left(\lambda_0(x) - \frac{\lambda_{n-1}(x)}{s_{n-1}(x)}s_0(x)\right)\right) \\
&= s_n(x)s_n(x+1)\left(\frac{\Delta\lambda_{n-1}(x) + \lambda_{n-1}(x+1)\lambda_0(x) + s_{n-1}(x+1)}{s_{n-1}(x+1)}\right. \\
&\quad \left. - \frac{\lambda_{n-1}(x)(\Delta s_{n-1}(x) + \lambda_{n-1}(x+1)s_0(x))}{s_{n-1}(x)s_{n-1}(x+1)}\right) \\
&= s_n(x)s_n(x+1)\left(\frac{\lambda_n(x)}{s_{n-1}(x+1)} - \frac{\lambda_{n-1}(x)s_n(x)}{s_{n-1}(x)s_{n-1}(x+1)}\right) \\
&= s_n(x)s_n(x+1)\left(\frac{s_{n-1}(x)\lambda_n(x) - \lambda_{n-1}(x)s_n(x)}{s_{n-1}(x)s_{n-1}(x+1)}\right) = 0.
\end{aligned}$$

This completes the proof. ■

At this stage we make the assumption that

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}, \tag{27}$$

holds for some n, hence for all the subsequent n's.

Theorem 3.3: *A solution of the difference equation*

$$\Delta^2 y(x) = \lambda_0(x)\Delta y(x) + s_0(x)y(x),$$

is given by

$$y(x) = \left(\prod_{i=x_0}^{x-1}\left[1 - \frac{s_{n-1}(i)}{\lambda_{n-1}(i)}\right]\right), \quad x = 0, 1, 2, \dots, \tag{28}$$

provided that

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)},$$

where $\lambda_n(x)$ and $s_n(x)$ are given by (26).

Proof: Assume that y is defined by (28). Then

$$\frac{\Delta y(x)}{y(x)} = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}. \tag{29}$$

Applying Δ to (29) and use the quotient rule (17) we conclude that

$$\frac{\Delta^2 y(x)}{y(x+1)} - \left(\frac{\Delta y(x)}{y(x)}\right)^2 \frac{y(x)}{y(x+1)} = -\frac{\Delta s_{n-1}(x)}{\lambda_{n-1}(x+1)} + \frac{s_{n-1}(x)\Delta\lambda_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)},$$

which is equivalent to

$$\Delta^2 y(x) - \left(\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right)^2 y(x) = \left(\frac{s_{n-1}(x)\Delta\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta s_{n-1}(x)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)}\right) y(x+1). \tag{30}$$

Using the recursive DAIM sequences (26) we find that

$$\begin{aligned} &s_{n-1}(x)\Delta\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta s_{n-1}(x) \\ &= -s_{n-1}(x)\lambda_{n-1}(x+1)\lambda_0(x) - s_{n-1}(x)s_{n-1}(x+1) \\ &\quad + \lambda_{n-1}(x)\lambda_{n-1}(x+1)s_0(x), \end{aligned} \tag{31}$$

Now Equation (30) becomes

$$\begin{aligned} \Delta^2 y(x) = &\left(s_0(x) - \frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)}\right)\Delta y(x) \\ &+ \left(s_0(x) - \frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)} + \left(\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right)^2\right) y(x), \end{aligned}$$

which can be written as

$$\begin{aligned}\Delta^2 y(x) &= \lambda_0(x)\Delta y(x) + s_0(x)y(x)\\ &\quad + \left(s_0(x) - \lambda_0(x) - \frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)}\right)\Delta y(x)\\ &\quad + \left(-\frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)} + \left(\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right)^2\right)y(x),\end{aligned}$$

Thus, to show that $\Delta^2 y(x) - \lambda_0(x)\Delta y(x) - s_0(x)y(x) = 0$, we need to show that

$$\begin{aligned}&\left(s_0(x) - \lambda_0(x) - \frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)}\right)\Delta y(x)\\ &\quad = -\left(-\frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)} + \left(\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right)^2\right)y(x).\end{aligned}$$

Using (30) we see that we need to show that

$$\begin{aligned}&\left(-\lambda_0(x) - \frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)} + s_0(x)\right)\Delta y(x)\\ &\quad + \left(\lambda_0(x) + \frac{s_{n-1}(x+1)}{\lambda_{n-1}(x+1)} - \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right)\Delta y(x) = 0,\end{aligned}$$

which is equivalent to showing that

$$s_0(x) - \frac{s_{n-1}(x)\lambda_0(x)}{\lambda_{n-1}(x)} - \frac{s_{n-1}(x)s_{n-1}(x+1)}{\lambda_{n-1}(x)\lambda_{n-1}(x+1)} + \frac{s_{n-1}(x+1)}{\lambda_{n-1}(x+1)} - \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = 0.$$

Multiply the above equality by $\lambda_{n-1}(x)\lambda_{n-1}(x+1)$ and apply (31) to reduce the problem to

$$s_{n-1}(x)\Delta\lambda_{n-1}(x) - \lambda_{n-1}(x)\Delta s_{n-1}(x) + s_{n-1}(x+1)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_{n-1}(x+1) = 0,$$

which is obviously true. ■

We now assume that there is an n such that (27) holds. In this case

$$\frac{\Delta^{n+2}y(x)}{\Delta^{n+1}y(x)} = \frac{\lambda_n(x)\Delta y(x) + s_n(x)y(x)}{\lambda_{n-1}(x)\Delta y(x) + s_{n-1}(x)y(x)} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}. \tag{32}$$

This implies

$$\Delta^{n+1}y(x) = \Delta^{n+1}y(0)\prod_{k=0}^{x-1}\left[1 + \frac{\lambda_n(k)}{\lambda_{n-1}(k)}\right]. \tag{33}$$

This is the exact analogue of Equation (2.10) in [5]. Note that (33) implies

$$\Delta^{n+1}y(x+m) = \Delta^{n+1}y(x)\prod_{k=0}^{m-1}\left[1 + \frac{\lambda_n(x+k)}{\lambda_{n-1}(x+k)}\right]. \tag{34}$$

Using Theorem 3.1 we find that the solution to the difference equation

$$\Delta^2 y(x) = \lambda_0(x)\Delta y(x) + s_0(x)y(x),$$

solves the first-order inhomogeneous difference equation

$$\Delta^{n+1}y(x)\prod_{k=0}^{m-1}\left[1+\frac{\lambda_n(x+k)}{\lambda_{n-1}(x+k)}\right] = \lambda_{n-1}(x+m)\Delta y(x+m) + s_{n-1}(x+m)y(x+m), \tag{35}$$

namely, for $m = 0, 1, 2, \ldots$,

$$\Delta y(x+m) + \frac{s_{n-1}(x+m)}{\lambda_{n-1}(x+m)}y(x+m) = \frac{\Delta^{n+1}y(x)}{\lambda_{n-1}(x+m)}\prod_{k=0}^{m-1}\left[1+\frac{\lambda_n(x+k)}{\lambda_{n-1}(x+k)}\right]. \tag{36}$$

Comparing this with (3) and (4) and replacing $\Delta^{n+1}y(x)$ by its value from (33) we see that the general solution, using $y(x) = y(x-m+m)$ is given by

$$\begin{aligned} y(x) &= C_2\prod_{i=n_0}^{x-1}\left(1-\frac{s_{n-1}(i)}{\lambda_{n-1}(i)}\right) \\ &\quad + C_1\sum_{i=n_0}^{x-1}\left(\prod_{\ell=i+1}^{x-1}\left(1-\frac{s_{n-1}(\ell)}{\lambda_{n-1}(\ell)}\right)\frac{\left(\prod_{j=n_0}^{i-m-1}\left(1+\frac{\lambda_n(j)}{\lambda_{n-1}(j)}\right)\right)}{\lambda_{n-1}(i)}\right. \\ &\quad \left.\prod_{k=0}^{m-1}\left[1+\frac{\lambda_n(i-m+k)}{\lambda_{n-1}(i-m+k)}\right]\right). \end{aligned} \tag{37}$$

Theorem 3.4: *The general solution to* (22) *is given by* (37), *where C_1 and C_2 are unit periodic functions provided that* (27) *is satisfied.*

Proof: The analysis before this theorem shows that (37) gives a solution of (22). So, we only need to show that the coefficients of C_1 and C_2, say $y_1(x)$ and $y_2(x)$ are linear independent. This holds if and only if the Casorati determinant

$$\begin{vmatrix} y_1(x) & y_1(x+1) \\ y_2(x) & y_2(x+1) \end{vmatrix}, \tag{38}$$

does not vanish, which is an easy exercise. ■

4. A criterion for polynomial solutions

The main results of this section are Theorems 4.1–4.2 which, respectively, give necessary, and sufficient conditions for a second order linear difference equation to have a polynomial solution.

Theorem 4.1: *If the second-order difference equation* $\Delta^2 y(x) = \lambda_0(x)\Delta y(x) + s_0(x)y(x)$ *has a polynomial solution of degree n, then*

$$s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x) = 0,$$

where

$$\lambda_n(x) = \Delta\lambda_{n-1}(x) + \lambda_{n-1}(x+1)\lambda_0(x) + s_{n-1}(x+1),$$
$$s_n(x) = \Delta s_{n-1}(x) + \lambda_{n-1}(x+1)s_0(x).$$

Proof: We apply (25) and the recursions in (26) to find that

$$s_n(x)\Delta^{n+1}y(x) = s_n(x)\lambda_{n-1}(x)\Delta y(x) + s_n(x)s_{n-1}(x)y(x),$$
$$s_{n-1}(x)\Delta^{n+2}y(x) = s_{n-1}(x)\lambda_n(x)\Delta y(x) + s_{n-1}(x)s_n(x)y(x), \tag{39}$$

which then yields

$$s_n(x)\Delta^{n+1}y(x) - s_{n-1}(x)\Delta^{n+2}y(x) = (s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x))\Delta y(x), \tag{40}$$

If $y(x)$ is a polynomial of degree n then $\Delta^{n+1}y(x) = \Delta^{n+2}y(x) = 0$ and the theorem follows. ■

The next theorem provides a converse to Theorem 4.1.

Theorem 4.2: *If* $s_n(x)\lambda_{n-1}(x) \neq 0$ *and* $\lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x) = 0$, *then the difference equation* $\Delta^2 y(x) = \lambda_0(x)\Delta y(x) + s_0(x)y(x)$ *has a polynomial solution whose degree is at most n.*

Proof: When $s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x) = 0$, Equation (40) reduces to

$$s_n(x)\Delta^{n+1}y(x) - s_{n-1}(x)\Delta^{n+2}y(x) = 0 \tag{41}$$

which yields

$$\begin{aligned} s_n(x)\Delta^{n+1}y(x) &= s_{n-1}(x)\left(\lambda_n(x)\Delta y(x) + s_n(x)y(x)\right) \\ &= s_{n-1}(x)y(x)\left(\lambda_n(x)\frac{\Delta y(x)}{y(x)} + s_n(x)\right). \end{aligned} \tag{42}$$

Let $y(x)$ be the solution given by (28) then apply (29) to establish

$$\begin{aligned} s_n(x)\Delta^{n+1}y(x) &= s_{n-1}(x)y(x)\left(-\lambda_n(x)\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} + s_n(x)\right) \\ &= \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}y(x)\left(\lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x)\right). \end{aligned}$$

Therefore

$$\Delta^{n+1}y(x) = \frac{s_{n-1}(x)}{s_n(x)\lambda_{n-1}(x)}y(x)\left(\lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x)\right) = 0.$$

This shows that $y(x)$ is a polynomial of degree at most n. ■

5. Examples

5.1. An equation of Euler type

Consider the equation

$$\Delta^2 y(x) = \frac{2(a-1)}{1+x}\Delta y(x) + \frac{a(1-a)}{x(1+x)}y(x). \tag{43}$$

Before applying DAIM to (100) we explain the relevance of Remark 1.1. If $y = x^n +$ lowerorderterms, then $x^2\Delta^2 y - n(n-1)x^n$ and $x\Delta y - nx^n$ are polynomials of degree at most $n-1$. Substituting $y = x^n +$ lowerorderterms in (100) and equating coefficients of x^n establishes the condition $n(n-1) = 2n(a-1) + a(1-a)$, which implies $a = n,\ n+1$. These are necessary conditions.

We now apply DAIM with

$$\lambda_0(x) = \frac{2(a-1)}{1+x}, \quad s_0(x) = \frac{a-a^2}{x(1+x)}. \tag{44}$$

From the DAIM sequences (26), we note that

$$\lambda_1(x) = \frac{3(a-2)(a-1)}{(1+x)(2+x)}, \quad s_1(x) = -\frac{2(a-2)(a-1)a}{x(1+x)(2+x)} \tag{45}$$

and after computing the first few λ_n's and s_n's we use induction to show that for arbitrary n, we have

$$\lambda_n(x) = \frac{(n+2)\prod_{k=0}^{n}(a-k-1)}{\prod_{k=0}^{n}(x+k+1)}, \quad s_n(x) = -\frac{(n+1)a\prod_{k=0}^{n}(a-k-1)}{\prod_{k=0}^{n+1}(x+k)}. \tag{46}$$

We then conclude that

$$\delta_n(x) = \lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)s_n(x) = -\frac{a(1-a)_n(1-a)_{n+1}}{(x)_{n+1}(x+1)_{n+1}}. \tag{47}$$

Thus $\delta_n(x) = 0$ if $a = n+1$. To construct the exact solution where $a = n+1$, we apply (28) and find that

$$y_n(x) = \left(\prod_{i=x_0}^{x-1}\left[1+\frac{n}{i}\right]\right) = \frac{(x)_n}{(x_0)_n}, \quad n = 0, 1, 2, \ldots. \tag{48}$$

To find a second independent solution, we shall use two different approaches, first using the second independent solution as given by Equation (37) with $a = n+1$, $m \equiv n = 1, 2, \ldots$,

$$y_2(x) = \sum_{i=n_0}^{x-1}\left(\prod_{\ell=i+1}^{x-1}\left(1-\frac{s_{n-1}(\ell)}{\lambda_{n-1}(\ell)}\right)\frac{\left(\prod_{j=n_0}^{i-m-1}\left(1+\frac{\lambda_n(j)}{\lambda_{n-1}(j)}\right)\right)}{\lambda_{n-1}(i)}\right.$$
$$\left.\prod_{k=0}^{m-1}\left[1+\frac{\lambda_n(i-m+k)}{\lambda_{n-1}(i-m+k)}\right]\right)$$

$$= \sum_{i=n_0}^{x-1} \frac{(-1)^{1+n} 2^{n_0-n} \Gamma(i+n+1)(n+1)_{x-1}}{n(n+1)\Gamma(x)(1-n)_{n-1}(n+1)_i}$$

$$\times \left(\frac{2(n+1)(1-n)_{n-1} \left(\frac{i-n+3}{2}\right)_n - (n+2)(1-n)_n \left(\frac{i-n+3}{2}\right)_{n-1}}{(n+1)(1-n)_{n-1} \left(\frac{i-n+3}{2}\right)_n} \right)^{n-n_0}$$

$$= \frac{(x-n_0)\Gamma(x+n)}{\Gamma(x)} = (x)_{n+1} + (n-n_0)(x)_n.$$

A second approach to find the other independent solution follows using the next lemma.

Lemma 5.1 ([20, Lemma 2, p. 3221]): *Let f and g be two linearly independent solutions of equation*

$$\Delta^n w(x) + a_{n-1}(x)\Delta^{n-1} w(x) + \cdots + a_1(x)\Delta w(x) + a_0(x) w(x) = 0 \tag{49}$$

Set $u = \Delta(f/g)$. *Then* $w = u(x)$ *satisfies*

$$\Delta^{n-1} w(x) + b_{n-1}(x)\Delta^{n-2} w(x) + \cdots + b_1(x)\Delta w(x) + b_0(x) w(x) = 0 \tag{50}$$

where

$$b_j(x) = \sum_{k=j+1}^{n} \binom{k}{j+1} a_k(x) \frac{\Delta^{k-j-1} g(x+j+1)}{g(x+n)}, \quad j = 0, 1, 2, \ldots, n-2. \tag{51}$$

Here we have, by convention, $a_n(x) = 1$.

For $n = 2$, the difference equation (49) reads

$$\Delta^2 w(x) + a_1(x)\Delta w(x) + a_0(x) w(x) = 0 \tag{52}$$

with $f(x)$ and $g(x)$ be two linearly independent solutions. Then $w = u(x) = \Delta(f/g)$ satisfies the first-order difference equation

$$\Delta w(x) + b_0(x) w(x) = 0 \tag{53}$$

where

$$b_0(x) = a_1(x) \frac{g(x+1)}{g(x+2)} + 2 \frac{\Delta g(x+1)}{g(x+2)} = 2 + (a_1(x) - 2) \frac{g(x+1)}{g(x+2)}, \tag{54}$$

for, by convention, $a_2(x) = 1$. The solution of the first order difference equation (53) is given by

$$w(x) = C_1 \prod_{j=n_0}^{x-1} (1 - b_0(j)) = C_1 \prod_{j=n_0}^{x-1} \left(-1 - \left(a_1(j) - 2\right) \frac{g(j+1)}{g(j+2)} \right) \tag{55}$$

and the second independent solution $f(x)$ follows by solving the first-order inhomogeneous difference equation

$$f(x+1) - \frac{g(x+1)}{g(x)} f(x) = C_1 g(x+1) \prod_{j=n_0}^{x-1} \left(\left(2 - a_1(j)\right) \frac{g(j+1)}{g(j+2)} - 1 \right) \tag{56}$$

The solution of the equation is easily found to be

$$f(x) = C_2 \left(\prod_{i=n_0}^{x-1} \frac{g(i+1)}{g(i)} \right) + C_1 \sum_{i=n_0}^{x-1} \left[\left(\prod_{\ell=i+1}^{x-1} \frac{g(\ell+1)}{g(\ell)} \right) g(i+1) \prod_{j=n_0}^{i-1} \left(\left(2 - a_1(j)\right) \frac{g(j+1)}{g(j+2)} - 1 \right) \right] \tag{57}$$

which resemble the general solution as given by (37). Thus, $a_1(x) = -2n/(1+x)$ and $g(x) = (x)_n$, it follow that

$$f(x) = C_2 \left(\prod_{i=n_0}^{x-1} \frac{i+n}{i} \right) + C_1 \sum_{i=n_0}^{x-1} \left[\left(\prod_{\ell=i+1}^{x-1} \frac{\ell+n}{\ell} \right) (i+1)_n \prod_{j=n_0}^{i-1} \left(\left(2 + \frac{2n}{1+j}\right) \frac{j+1}{j+1+n} - 1 \right) \right] \tag{58}$$

Straightforward computation shows that

$$f(x) = C(x)_n + B(x)_{n+1}, \tag{59}$$

where C and B are unit periodic functions as expected and easily confirmed by direct substitution.

5.2. Difference equation for dual polynomials

Let $\{Q_n(x)\}$ be a sequence of discrete orthogonal polynomials and let

$$\sum_{j=0}^{\infty} Q_m(x_j) Q_n(x_j) w_j = \delta_{m,n}/u_n. \tag{60}$$

Thus the rows of the matrix whose (i,j) element is $\{Q_i(x_j)\sqrt{u_i w_j}\}$, $i,j = 0,1,\dots$ are orthonormal vectors. The associativity of matrix multiplication then implies that this matrix is an orthogonal matrix. This forces the columns to be orthonormal vectors, that is

$$\sum_{n=0}^{\infty} Q_n(x_i) Q_n(x_j) u_n = \delta_{i,j}/w_j. \tag{61}$$

A birth and death process [17] with birth rates $\{\beta(n)\}$ and death rates $\{d(n)\}$ generates a sequence of orthogonal polynomials $\{Q_n(x)\}$. The initial values are $Q_0(x) = 1, Q_1(x) = (b(0) + d(0) - x)/b(0)$ and the recurrence relation

$$-xQ_n(x) = b(n)Q_{n+1}(x) + d(n)Q_{n-1}(x) - [b(n) + d(n)]Q_n(x), \quad n > 0. \tag{62}$$

If $\{Q_n(x)\}$ is orthogonal with respect to a discrete measure then the dual polynomials $\{Q_n(x_j) : j = 0, 1, \dots\}$, where now the variable is n and the degree is j is called the polynomial dual to $\{Q_n(x)\}$. There are many instances of this in the Askey scheme [18]. The bispectral problem of Duistermaat and Grünbaum [6] is also related to this phenomenon. In such cases the dual polynomials will satisfy the difference equation

$$\xi y(x) = b(x)y(x+1) + d(x)y(x-1) - [b(x) + d(x)]y(x). \tag{63}$$

In other words

$$b(x+1)\Delta^2 y(x) + [b(x+1) - d(x+1) - \xi]\Delta y(x) - \xi y(x) = 0. \tag{64}$$

The case of birth and death process polynomials when $b(x)$ and $d(x)$ are polynomials of degree at most 2 and $b(x) - d(x)$ is of degree at most 1 was studied in [14], where their orthogonality measure was also constructed. Their dual polynomials will then satisfy the hypergeometric difference equation

$$(a_2x^2 + a_1x + a_0)\Delta^2 y(x) + (b_1x + b_0)\Delta y(x) - ky(x) = 0, \tag{65}$$

This equation may also be considered as a difference analogue of the hypergeometric difference equation.

$$\lambda_0(x) = -\frac{b_1x + b_0}{a_2x^2 + a_1x + a_0}, \quad s_0(x) = \frac{k}{a_2x^2 + a_1x + a_0}, \tag{66}$$

it follow, by the recursive evaluation of the DAIM sequence, that the termination condition $\delta_n(x) = \lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)s_n(x), n = 1, 2, \ldots$ yields

$$\begin{aligned}
\delta_1(x) &= \frac{b_1 - k}{a_0 + (1+x)(a_1 + a_2(1+x))}\delta_0(x)\\
\delta_2(x) &= \frac{2a_2 + 2b_1 - k}{a_0 + (2+x)(a_1 + a_2(2+x))}\delta_1(x)\\
\delta_3(x) &= \frac{6a_2 + 3b_1 - k}{a_0 + (3+x)(a_1 + a_2(3+x))}\delta_2(x)\\
\delta_4(x) &= \frac{12a_2 + 4b_1 - k}{a_0 + (4+x)(a_1 + a_2(4+x))}\delta_3(x).
\end{aligned}$$

For arbitrary n, it is not difficult to show that, for $n = 1, 2, \ldots$,

$$\begin{aligned}
\delta_n(x) &= \frac{n(n-1)a_2 + nb_1 - k}{a_0 + (n+x)(a_1 + a_2(n+x))}\delta_{n-1}(x)\\
&= \frac{\prod_{j=0}^n j(j-1)a_2 + jb_1 - k}{\prod_{j=0}^n a_0 + (j+x)(a_1 + a_2(j+x))}\delta_0(x)
\end{aligned} \tag{67}$$

where $\delta_0(x) = s_0(x)$ given $\lambda_{-1}(x) = -1, s_{-1}(x) = 0$. Clearly, for $\delta_{n-1}(x) \neq 0$, $\delta_n(x) = 0$ only if

$$k = n(n-1)a_2 + n\,b_1, \quad n = 1, 2, \ldots. \tag{68}$$

In this case, the polynomial solutions of the difference equation

$$\Delta^2 y(x) = -\frac{b_1x + b_0}{a_2x^2 + a_1x + a_0}\Delta y(x) + \frac{n(n-1)a_2 + n\,b_1}{a_2x^2 + a_1x + a_0}y(x), \tag{69}$$

are given as

- For $n = 0$, $y_0(x) = 1$.
- For $n = 1$,

$$y_1(x) = \prod_{i=x_0}^{x-1}\left[1 - \frac{s_0(x)}{\lambda_0(x)}\right] = x + \frac{b_0}{b_1}.$$

- For $n = 2$,

$$y_2(x) = \prod_{i=x_0}^{x-1}\left[1 - \frac{s_1(x)}{\lambda_1(x)}\right] = x^2 + \frac{(2a_1 + 2b_0 + b_1)}{(2a_2 + b_1)}x + \frac{(a_0(2a_2 + b_1) + b_0(a_1 + a_2 + b_0 + b_1))}{((a_2 + b_1)(2a_2 + b_1))}.$$

- For $n = 3$,

$$\begin{aligned} y_3(x) &= x^3 + \frac{3(2a_1 + 2a_2 + b_0 + b_1)}{(4a_2 + b_1)}x^2 \\ &+ \frac{(6a_1^2 + 12a_2b_0 + 3b_0^2 + 5a_2b_1 + 6b_0b_1 + 2b_1^2 + 3a_0(4a_2 + b_1) + 9a_1(2a_2 + b_0 + b_1))}{(3a_2 + b_1)(4a_2 + b_1)}x \\ &+ (a_0(36a_2^2 + 10a_2b_0 + 24a_2b_1 + 3b_0b_1 + 4b_1^2 + 4a_1(3a_2 + b_1)) \\ &+ b_0(2a_1^2 + 10a_2^2 + 7a_2b_0 + b_0^2 + 9a_2b_1 + 3b_0b_1 + 2b_1^2 \\ &+ a_1(12a_2 + 3b_0 + 5b_1))/((2a_2 + b_1)(3a_2 + b_1)(4a_2 + b_1)). \end{aligned}$$

and so on for higher order.

As special cases of the hypergeometric difference equation (65) are the Meixner difference equation

$$\Delta^2 y(x) = -\frac{(\mu - 1)(x - n + 1) + \mu\delta}{\mu(x + \delta + 1)}\Delta y(x) - \frac{k}{\mu(x + \delta + 1)}y(x), \tag{70}$$

and the Hermite difference equation

$$\Delta^2 y(x) = (a\,x + b)\Delta y(x) + \gamma\, y(x). \tag{71}$$

6. *q*-Asymptotic iteration method (*q*AIM)

We consider the linear second-order q-difference equation

$$D_q^2 y(x) = \lambda_0(x) D_q y(x) + s_0(x) y(x). \tag{72}$$

In general, we have

$$D_q^{n+2} y(x) = \lambda_n(x) D_q y(x) + s_n(x) y(x), \tag{73}$$

where the functions $\lambda_n(x)$ and $s_n(x)$ are generated by

$$\lambda_n(x) = D_q\lambda_{n-1}(x) + \lambda_{n-1}(qx)\lambda_0(x) + s_{n-1}(qx), \quad s_n(x) = D_q s_{n-1}(x) + \lambda_{n-1}(qx)s_0(x). \tag{74}$$

If the termination condition

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}. \tag{75}$$

holds for some n then

$$\frac{D_q^{n+2}y(x)}{D_q^{n+1}y(x)} = \frac{\lambda_n(x)D_qy(x) + s_n(x)y(x)}{\lambda_{n-1}(x)D_qy(x) + s_{n-1}(x)y(x)} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}. \tag{76}$$

Equation (76) can be written as

$$D_q\left(D_q^{n+1}y(x)\right) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)}D_q^{n+1}y(x) \tag{77}$$

This is a first-order q-difference equation in $D_q^{n+1}y(x)$ and according to (18)–(19) its solution is

$$D_q^{n+1}y(x) = D_q^{n+1}y(0)\prod_{k=0}^{\infty}\left[1-(1-q)q^kx\frac{\lambda_n(q^kx)}{\lambda_{n-1}(q^kx)}\right]^{-1}. \tag{78}$$

The infinite product will converge if the ratio $\lambda_n(x)/\lambda_{n-1}(x)$ is bounded in a neighbourhood of $x = 0$ in the complex plane. On the other hand (73) implies

$$\lambda_{n-1}(x)D_qy(x) + s_{n-1}(x)y(x) = \frac{D_q^{n+1}y(0)}{\prod_{k=0}^{\infty}\left[1-(1-q)q^kx\frac{\lambda_n(q^kx)}{\lambda_{n-1}(q^kx)}\right]} \tag{79}$$

or equivalently

$$D_qy(x) = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}y(x) + \frac{D_q^{n+1}y(0)}{\lambda_{n-1}(x)}\prod_{k=0}^{\infty}\left[1-(1-q)q^kx\frac{\lambda_n(q^kx)}{\lambda_{n-1}(q^kx)}\right]^{-1}. \tag{80}$$

In view of (20)–(21) the solution of the original second-order q-difference equation (72) is given by

$$y(x) \tag{1}$$

$$= \frac{y(0)}{\prod_{k=0}^{\infty}\left[1+(1-q)q^kx\frac{s_{n-1}(q^kx)}{\lambda_{n-1}(q^kx)}\right]}$$

$$+ D_q^{n+1}y(0)\sum_{k=0}^{\infty}\frac{\frac{(1-q)q^kx}{\lambda_{n-1}(q^kx)}}{\prod_{i=0}^{\infty}\left[1-(1-q)q^{i+k}x\frac{\lambda_n(q^{i+k}x)}{\lambda_{n-1}(q^{i+k}x)}\right]\prod_{j=0}^{k}\left[1-(1-q)q^jx\frac{s_{n-1}(q^jx)}{\lambda_{n-1}(q^jx)}\right]}. \tag{81}$$

It is known that y_1 and y_2 are linearly independent if and only if the determinant

$$\begin{vmatrix} y_1(x) & D_qy_1(x) \\ y_2(x) & D_qy_2(x) \end{vmatrix} \neq 0, \tag{82}$$

for all x in the domain of definition. It is easy to see that this is case with the two solutions given above.

7. Implementation and examples

Our first example is the q-Laguerre polynomials, [18, p. 109]. They satisfy the q-Difference equation:

$$\left(1+q^{\eta}+q^{\eta+n}x\right)y(x)=q^{\eta}(1+x)y(qx)+y(q^{-1}x), \tag{83}$$

It is easy to write this equation in the form

$$D_q^2y(x)=\left(\frac{q^{-1-\eta}-1-(1+q-q^n)x}{(q-1)x(1+qx)}\right)D_qy(x)+\left(\frac{q^n-1}{(q-1)^2x(1+qx)}\right)y(x), \tag{84}$$

with

$$\lambda_0(x)=\frac{q^{-1-\eta}-1-(1+q-q^n)x}{(q-1)x(1+qx)},\quad s_0(x)=\frac{q^n-1}{(q-1)^2x(1+qx)} \tag{85}$$

Using (77) and the definition

$$\delta_m(x)=\lambda_m(x)s_{m-1}(x)-\lambda_{m-1}(x)s_m(x),\quad m=1,2,\ldots \tag{86}$$

it follow that

$$\begin{aligned}
\delta_1&=\frac{(q-q^n)(q^n-1)}{x^2(qx+1)(q^2x+1)(q-1)^4},\\
\delta_2&=\frac{(q-q^n)(q^2-q^n)(q^n-1)}{x^3(qx+1)(q^2x+1)(q^3x+1)(q-1)^6},\\
\delta_3&=\frac{(q-q^n)(q^2-q^n)(q^3-q^n)(q^n-1)}{x^4(qx+1)(q^2x+1)(q^3x+1)(q^4x+1)(q-1)^8},\\
\delta_4&=\frac{(q-q^n)(q^2-q^n)(q^3-q^n)(q^4-q^n)(q^n-1)}{x^5(qx+1)(q^2x+1)(q^3x+1)(q^4x+1)(q^5x+1)(q-1)^{10}},\\
\delta_5&=\frac{(q-q^n)(q^2-q^n)(q^3-q^n)(q^4-q^n)(q^5-q^n)(q^n-1)}{x^6(qx+1)(q^2x+1)(q^3x+1)(q^4x+1)(q^5x+1)(q^6x+1)(q-1)^{12}}.
\end{aligned}$$

In general we observe the pattern

$$\delta_{m+1}=\frac{q^{m+1}-q^n}{(q-1)^2x(1+q^{m+2})}\delta_m,\quad m=0,1,2,\ldots \tag{87}$$

which has been tested up to $m=15$. Based on this we conclude that $\delta_m=0$ if and only if $m=n$. For an exact solution, we use the following expression:

$$y_n(x)=\frac{y_n(0)}{\prod_{k=0}^{\infty}\left[1+(1-q)q^kx\frac{s_{n-1}(q^kx)}{\lambda_{n-1}(q^kx)}\right]}. \tag{88}$$

For example, the polynomial solution of degree 5 is

$$
\begin{aligned}
y_5(x) &= y_5(0)\prod_{k=0}^{\infty}\left[1+(1-q)q^k x\frac{s_4(q^k x)}{\lambda_4(q^k x)}\right]^{-1}\\
&= y_5(0)\Bigg(1+\frac{q^{1+\eta}\left(1-q^5\right)}{(1-q)(q^{1+\eta}-1)}x+\frac{q^{4+2\eta}(1+q^2)(1-q^5))}{(1-q)(q^{1+\eta}-1)(q^{2+\eta}-1)}x^2\\
&\quad+\frac{q^{9+3\eta}\left(1+q^2\right)\left(1-q^5\right)}{(1-q)\left(q^{1+\eta}-1\right)\left(q^{2+\eta}-1\right)\left(q^{3+\eta}-1\right)}x^3\\
&\quad+\frac{q^{4(4+\eta)}\left(1-q^5\right)}{(1-q)\left(q^{1+\eta}-1\right)\left(q^{2+\eta}-1\right)\left(q^{3+\eta}-1\right)\left(q^{4+\eta}-1\right)}x^4\\
&\quad+\frac{q^{5(5+\eta)}}{\left(q^{1+\eta}-1\right)\left(q^{2+\eta}-1\right)\left(q^{3+\eta}-1\right)\left(q^{4+\eta}-1\right)\left(q^{5+\eta}-1\right)}x^5\Bigg).
\end{aligned} \tag{89}
$$

More importantly we can also write down a second solution to the q-difference equation. It is know that the second solution is related to the function of the second kind, see [12,13].

Our second example is the Al-Salam-Carlitz polynomials $\{U_n(x)\}$, [12,18]. Their q-Difference equation is

$$
\begin{aligned}
aq^{n-1}y(q^2 x) &= \left(aq^{-1+n}+aq^n-(1+a)q^{1+n}x+q^2x^2\right)y(qx)\\
&\quad-q^n(1-qx)(a-qx)y(x).
\end{aligned} \tag{90}
$$

Thus

$$
D_q^2 y(x) = \left(\frac{q+aq-q^{2-n}x}{a-aq}\right)D_q y(x)-\frac{q^{2-n}\left(-1+q^n\right)}{a(-1+q)^2}y(x) \tag{91}
$$

The termination condition $\delta_n(x)=\lambda_n(x)s_{n-1}(x)-s_n(x)\lambda_{n-1}(x)\equiv 0$, $n=1,2\ldots$ where $\{\lambda_n(x)\}$ and $\{s_n\}$ satisfy, see (74),

$$
\begin{aligned}
\delta_1(x) &= \frac{q^{2(2-n)}(q^n-1)(q-q^n)}{a^2(q-1)^4} = \frac{q^{2-n}(q-q^n)}{a(q-1)^2}\delta_0(x),\\
\delta_2(x) &= \frac{q^{3(2-n)}(q^n-1)(q-q^n)(q^2-q^n)}{a^3(q-1)^6} = \frac{q^{2-n}(q^2-q^n)}{a(q-1)^2}\delta_1(x),\\
\delta_3(x) &= \frac{q^{4(2-n)}(q^n-1)(q-q^n)(q^2-q^n)(q^3-q^n)}{a^4(q-1)^8} = \frac{q^{2-n}(q^3-q^n)}{a(q-1)^2}\delta_2(x),\\
\delta_4(x) &= \frac{q^{5(2-n)}(q^n-1)(q-q^n)(q^2-q^n)(q^3-q^n)(q^4-q^n)}{a^5(q-1)^{10}} = \frac{q^{2-n}(q^4-q^n)}{a(q-1)^2}\delta_3(x),\\
\delta_5(x) &= \frac{q^{6(2-n)}(q^n-1)(q-q^n)(q^2-q^n)(q^3-q^n)(q^4-q^n)(q^5-q^n)}{a^6(q-1)^{12}}\\
&= \frac{q^{2-n}(q^5-q^n)}{a(q-1)^2}\delta_4(x).
\end{aligned}
$$

We may then observe the pattern

$$\delta_{m+1}(x) = \frac{q^{2-n}(q^{m+1} - q^n)}{a(q-1)^2}\delta_m, \quad m = 1, 2, \dots. \tag{92}$$

We verified this pattern up to $m = 15$. Thus the smallest m which makes $\delta_m(x) = 0$ is $m = n$. The polynomials solution is then given by (88). For example the polynomial of order five is given by

$$\begin{aligned}
&y_5(x)/y_5(0) \\
&= 1 - \frac{\begin{array}{c}(1+q+q^2+q^3+q^4)((1+a^4)q^4 + aq(1+q)(1+q^2)(1+a^2) \\ +a^2(1+q^2)(1+q+q^2))\end{array}}{\begin{array}{c}(1+a)q^2(q^6+a^4q^6+aq^2(1+q)(1+q^2) \\ +a^3q^2(1+q)(1+q^2)+a^2(1+q^2)(1+q+q^4))\end{array}}x \\
&\quad + \frac{(1+q^2)(a+aq+(1+a^2)q^2)(1+q+q^2+q^3+q^4)}{\begin{array}{c}q^3(q^6+a^4q^6+aq^2(1+q)(1+q^2)+a^3q^2(1+q)(1+q^2) \\ +a^2(1+q^2)(1+q+q^4))\end{array}}x^2 \\
&\quad - \frac{(a+(1+a+a^2)q)(1+q^2)(1+q+q^2+q^3+q^4)}{\begin{array}{c}(1+a)q^4(q^6+a^4q^6+aq^2(1+q)(1+q^2) \\ +a^3q^2(1+q)(1+q^2)+a^2(1+q^2)(1+q+q^4))\end{array}}x^3 \\
&\quad + \frac{1+q+q^2+q^3+q^4}{\begin{array}{c}q^4(q^6+a^4q^6+aq^2(1+q)(1+q^2)+a^3q^2(1+q)(1+q^2) \\ +a^2(1+q^2)(1+q+q^4))\end{array}}x^4 \\
&\quad \cdot - \frac{x^5}{\begin{array}{c}(1+a)q^4(q^6+a^4q^6+aq^2(1+q)(1+q^2)+a^3q^2(1+q)(1+q^2) \\ +a^2(1+q^2)(1+q+q^4)\end{array}}.
\end{aligned}$$

Our third example is the Stieltjes-Wigert q-difference equation, [19, p. 116]. The q-Difference equation satisfied by the Stieltjes–Wigert polynomials is

$$-x(1-q^n)y(x) = xy(qx) - (1+x)y(x) + y(q^{-1}x), \tag{93}$$

which has the equivalent form

$$D_q^2 y(x) = \left(\frac{1 - q\left(1 + q - q^n\right)x}{(q-1)q^2x^2}\right) D_q y(x) + \frac{q^n - 1}{(q-1)^2 q x^2} y(x). \tag{94}$$

Using the recursion (74) with

$$\lambda_0(x) = \left(\frac{1 - q\left(1 + q - q^n\right)x}{(q-1)q^2x^2}\right), \quad s_0(x) = \frac{q^n - 1}{(q-1)^2 q x^2}, \tag{95}$$

we find that

$$\delta_1(x) = \frac{(q^n-1)(q^n-q)}{q^3x^4(q-1)^4} = \frac{q^n-q}{q^2x^2(q-1)^2}\delta_0(x),$$

$$\delta_2(x) = \frac{(q^n-1)(q^n-q)(q^n-q^2)}{q^6x^6(q-1)^6} = \frac{q^n-q^2}{q^3x^2(q-1)^2}\delta_1(x),$$

$$\delta_3(x) = \frac{(q^n-1)(q^n-q)(q^n-q^2)(q^n-q^3)}{q^{10}x^8(q-1)^8} = \frac{q^n-q^3}{q^4x^2(q-1)^2}\delta_2(x),$$

$$\delta_4(x) = \frac{(q^n-1)(q^n-q)(q^n-q^2)(q^n-q^3)(q^n-q^4)}{q^{15}x^{10}(q-1)^{10}} = \frac{q^n-q^4}{q^5x^2(q-1)^2}\delta_3(x),$$

$$\delta_5(x) = \frac{(q^n-1)(q^n-q)(q^n-q^2)(q^n-q^3)(q^n-q^4)(q^n-q^5)}{q^{21}x^{12}(q-1)^{12}} = \frac{q^n-q^5}{q^6x^2(q-1)^2}\delta_4(x).$$

This suggests the pattern

$$\delta_{m+1} = \frac{q^n-q^{m+1}}{q^{m+2}x^2(q-1)^2}\delta_m(x), \quad m = 0, 1, 2, \dots. \tag{96}$$

Again, we verified this up to $m = 15$. Thus the smallest m for which $\delta_m(x) = 0$ is $m = n$. The polynomial solutions are then given by (88). The fifth-order polynomial solution is given by

$$\begin{aligned} y_5(x) = y_5(0)(1 &- \left(1+q+q^2+q^3+q^4\right)(qx) \\ &+ \left(1+q^2\right)\left(1+q+q^2+q^3+q^4\right)(q^2x)^2 \\ &- \left(1+q^2\right)\left(1+q+q^2+q^3+q^4\right)(q^3x)^3 \\ &+ \left(1+q+q^2+q^3+q^4\right)(q^4x)^4 - (q^5x)^5. \end{aligned}$$

This can written in the form

$$\begin{aligned} \frac{y_5(x)}{y_5(0)} &= 1 + \frac{(1-q^5)}{1-q)}q(-x) + \frac{(1-q^5)(1-q^4)}{(1-q)(1-q^2)}q^4(-x)^2 \\ &+ \frac{(1-q^5)(1-q^4)}{(1-q)(1-q^2)}q^9(-x)^3 + \frac{(1-q^5)}{(1-q)}q^{16}(-x)^4 + q^{25}(-x)^5. \end{aligned} \tag{97}$$

From this pattern the following pattern is clear

$$\frac{y_n(x)}{y_n(0)} = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}(-1)^k q^{k^2} x^k, \tag{98}$$

which can then be proved rigorously.

Remark 7.1: It is important to note that it is not surprising that $\delta_m(x)$ for the q-Laguerre $\{L_n^{(\eta)}(x;q)\}$ and the Stieltjes–Wigert polynomials $\{S_n(x;q)\}$ are almost identical. The reason is that the part of $\delta_n(x)$ for the q-Laguerre polynomials which vanishes does not depend on η, and $S_n(x;q)$ and $L_n^{(\eta)}(x;q)$

$$S_n(x;q) = \lim_{\eta\to\infty} L_n^{(\eta)}(xq^{-\eta};q) \tag{99}$$

8. Limitations of DAIM and *q*-AIM

In this section we show the limitations of the both DAIM and q-AIM by applying it to the case of linear second-order difference equation with constant coefficients. The case of linear second-order differential equation with constant coefficients is similar. Consider the difference equation

$$\Delta^2 y(x) = a\,\Delta y(x) + b\,y(x) \tag{100}$$

where $\lambda_0(x) \equiv a$ and $s_0(x) \equiv b$ are polynomials in a and b. Therefore, the DAIM sequences (26) yields

$$\lambda_1(x) = a^2 + b, \quad s_1(x) = ab, \quad \lambda_2(x) = a(a^2 + 2b), \quad s_2(x) = b(a^2 + b)..$$

In the present case the recurrence relations (26) become

$$\lambda_n = \lambda_{n-1}\lambda_0 + s_{n-1}, \quad s_n = \lambda_{n-1}s_0. \tag{101}$$

it follows that λ_n and s_n are polynomials in a and b of total degree $n+1$. The termination condition

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} \quad \text{implies} \quad \frac{\lambda_{n-1}s_0}{\lambda_{n-1}\lambda_0 + s_{n-1}} = \frac{s_{n-1}}{\lambda_{n-1}},$$

which leads to the quadratic equation

$$\left(\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right)^2 + \lambda_0(x)\left(\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}\right) - s_0(x) = 0$$

with solutions

$$\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \frac{-a \pm \sqrt{a^2 + 4b}}{2}. \tag{102}$$

It is now clear that there is no n for which (102) holds because its left-hand side is a rational function in a and b but its right-hand side is an algebraic non-rational function. What is surprising is that the method nevertheless gives the correct answer. Indeed this gives the solutions

$$\begin{aligned} y_+(x) &= \prod_{i=n_0}^{x-1}\left[1 - \frac{s_{x-1}(i)}{\lambda_{x-1}(i)}\right] = C_1\left[1 + \frac{a - \sqrt{a^2 + 4b}}{2}\right]^x, \\ y_-(x) &= \prod_{i=n_0}^{x-1}\left[1 - \frac{s_{x-1}(i)}{\lambda_{x-1}(i)}\right] = C_2\left[1 + \frac{a + \sqrt{a^2 + 4b}}{2}\right]^x. \end{aligned} \tag{103}$$

Surprisingly, this is the correct answer, [16,22].

One is tempted to use (101) to get

$$\frac{s_n}{\lambda_n} = \frac{b}{a + \frac{s_{n-1}}{\lambda_{n-1}}},$$

which when iterated leads to the continued fraction [15]

$$\frac{s_n}{\lambda_n} = \frac{b}{a}\frac{}{+}\frac{b}{a}\frac{}{+}\cdots.$$

Here again we face issues of rigour because the above continued fraction is a periodic continued fraction and will converge to a unique value involving the minimal solution, via Pincherle's theorem [15]. So even formally we get only one solution.

There is also inherent inconsistency in applying AIM, DAIM, or q-AIM to equations with constant coefficients. In all cases it has been been proved that the terminating condition holds if and only if the equation in question has a polynomial solution. This automatically excludes all equations with constant coefficients, except trivial ones like $y'' = 0, \Delta^2 y(x) = 0, D_q^2 y(x) = 0$. This also invalidates the application of AIM to general Euler equations of the type

$$x^2 y''(x) + a\,x\,y'(x) + b\,y(x) = 0,$$

What is a surprise is that this invalid applications of the AIM, DAIM, or q-AIM technique give the correct answers.

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