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Pricing Multivariate Financial Derivatives Using Polynomial Approximations

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Contents

Introduction

Since the early days of the theory of option pricing, pioneered by Black, Scholes and Merton ([3] and [11]) the use of complex financial derivatives has increased tremendously. Finding prices of financial derivatives accurately and efficiently is an important problem for the financial industry for obvious reasons. It is also a challenging mathematical problem as we do not have explicit expressions for option prices in many situations, and the use of numerical methods is almost unavoidable.

In the available literature there are very general methods to price financial derivatives. Deriving and solving the corresponding pricing PDE can be used in many instances (see [6] for instance) but the numerical methods to solve PDEs are not efficient as the dimension of the problem increases. Monte Carlo methods are also popular (see [8]) and do not suffer from "curse of dimensionality" issues, but they can still be slow, and require many simulations to get pricing results within the desired accuracy.

Closed-form accurate approximations of option prices are preferable than other computationally expensive numerical methods. In the case of spread options (that depend on two assets) some initial works in this direction are [10], [4] and [5].

This honours project concerns the pricing of multivariate financial instruments. The objective of this work is to develop a pricing methodology that is accurate, computationally efficient, and that admits generalization to price derivatives that depend on an arbitrary number of assets.

We use polynomial approximations to derive our prices. The use of polynomials to obtain approximated option prices under similar models has been explored before. In [1], Taylor polynomials are used to price spread options. In [12] approximations based on Chebyshev polynomials are used to price basket options. Very recently, orthogonal polynomial expansions have been used to price European options under stochastic volatility models in [2]. Our approximation is based on polynomials (in some cases of several variables) that satisfy a least squares criteria. As far as we know, this is the first attempt to use polynomials obtained in this way to price multivariate financial derivatives.

Using a least squares [7] criteria to find an approximating polynomial offers some advantages compared to other approximating procedures. For example, Taylor polynomials can approximate functions locally very well, but they are not generally good to approximate functions on a large domain. Using least squares to approximate functions is a well known approximating method that applies in general inner product spaces so we can clearly use this notion to approximate polynomials in several variables.

In Chapter 1 we introduce some basic concepts needed in this thesis. These are brief overviews of the ideas and for detailed sources readers can refer to the books in the bibliography section at the end of the thesis. In this chapter we first introduce the Brownian motion in both one-dimensional and multi-dimensional cases. Then we introduce the asset price under the Black-Scholes model for both single-variate and multi-variate cases. In the last section we look at the option price as expected value of payoff, as well as pricing European call and put options with the Black-Scholes formula.

In Chapter 2 we introduce the pricing of bivariate derivatives. We first talk about the distribution of the conditional normal random variables, and by comparing the distribution with asset price expression we modify the Black-Scholes formula to price bivariate options. The modified Black-Scholes formula is then approximated by a polynomial function using the least square criteria, and the price is approximated by the expected value of the approximating function. At the end we obtain some numerical results and the results is compared to the Monte Carlo Methods

In Chapter 3 we introduce the pricing of trivariate derivatives. We start with the distribution of trivairate conditional normal random vairiable, then we compare the distribution with asset price expression to obtain a modified Black-Scholes formula to price the trivariate option. Then we use least square approximation method to find a approximation function for the modified Black-Scholes formula, as well as the approximated pricing as the expected value of approximating function.

At the end we obtain some numerical results and the results is compared to the Monte Carlo Methods

Chapter 1

Preliminaries

This chapter contains some critical preliminaries used throughout the thesis to make this thesis as self-contained as possible. However, these are only very brief summaries of each topic, future references on these topics are provided in the Appendix if one is interested in a more detailed explanation of each individual topic. In this chapter we will introduce Brownian motions, payoff of options, option prices as expected value of payoff, the Black-Scholes Model and Monte Carlo method for option pricing.

1.1 Brownian Motion

A one-dimensional Brownian motion (Wiener Process) on the interval $[0, T]$ is a stochastic process $\{W(t), 0 \leq t \leq T\}$ that has the following properties:

1. $W(0) = 0$.
2. $W(t)$ is a continuous function of t, with probability 1.
3. W has independent increments, $W(t+s) - W(t)$ is independent of $W(r)$ for $r \leq t$.
4. $W(t) - W(s) \sim N(0, t-s)$ for any $0 \leq s \leq t \leq T$.

We can also have d-dimensional Brownian motions , let $\mathbf{W}(t) = (W_1(t), W_2(t), \cdots, W_d(t))^T$, each component is itself a one-dimensional Brownian motion and properties 1 to 3 above are also satisfied for the multi-dimensional Brownian motion. Property 4 for the multi-dimensional Brownian motion changes to $\mathbf{W}(t)-\mathbf{W}(s) \sim N(0,(t-s)\Sigma)$ where

$$\Sigma = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,d} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,d} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \rho_{d,1} & \rho_{d,2} & \cdots & 1 \end{bmatrix}$$

the term $\rho_{i,j}$ is the correlation coefficient of $W_i(T)$ and $W_j(T)$, and $i, j = 1, 2, \cdots, d$.

For multivariate Brownian motion, we have $\mathbf{W}(T) = (W_1(T), W_2(T), \cdots, W_d(T))^T$ which follows the multivariate normal distribution $\mathbf{W}(T) \sim N(\mu, \Sigma)$. Therefore

$$\mathbf{W}(T) \sim N(0, T\Sigma) \tag{1.1}$$

Classical references on Brownian motion are [9] and [13].

1.2 Black-Scholes Model

Under the Black-Scholes model we have an expression that describes the movement of stock prices, and uses the Brownian motion as the uncertainty source in this way

$$S(t) = S(0)e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma W(t)}, \text{ for } t \in [0,T] \tag{1.2}$$

where $S(0)$ is the spot price at the beginning of the contract, r is the risk-free interest rate, δ is the continuous dividend yield, σ is the volatility and T is a time horizon.

Expression (1.2) implies that

$$\ln(S(T)) = \ln(S(0)) + \left(r - \delta - \frac{1}{2}\sigma^2\right)T + \sigma W(T). \tag{1.3}$$

Notice that $W(T) \sim N(0,T)$, we get $\ln(S(T))$ is also normally distributed. Moreover, the distribution of $\ln(S(T))$ satisfies that

$$E(\ln(S(T))) = \ln(S(0)) + (r - \delta - \frac{1}{2}\sigma^2)T, \qquad Var(\ln(S(T))) = \sigma^2 T \tag{1.4}$$

A multi-variate Black-Scholes $S(t) = (S_1(t), S_2(t), \cdots, S_d(t))^T$ is a process such that for all $i = 1, 2, \cdots, d$ we have

$$S_i(t) = S_i(0)e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma W_i(t)} \tag{1.5}$$

where $\mathbf{W}(t) = (W_1(t), W_2(t), \cdots, W_d(t))^T$ is a d-dimensional Brownian motion.

1.3 Multivariate Options

The European call and put options mentioned in the previous section only depend on a single asset. However, this thesis is focused on the pricing of multivariate financial derivatives. In this section, we are going to introduce some examples of options that depend on several assets

A spread option is an option that depends on two assets, the value of a spread option is derived from the difference of the values of its assets. For example, a spread option with strike price K and spot prices at expiration $S_1(T)$ and $S_2(T)$ has the following payoff,

$$\text{payoff} = max(0,\ S_1(T) - S_2(T) - K). \tag{1.6}$$

Similarly, a three-asset spread option depends on three assets and the value is determined by the difference of the value of three assets. In this case the payoff is

$$\text{payoff} = max(0,\ S_1(T) - S_2(T) - S_3(T) - K).$$

Another example of multivariate option is the basket option. In a basket option, the value is determined by the weighted sum or the average of its underlying assets

$$\text{payoff} = max(0, \ \sum \alpha_i S_i(T) - K)$$

where α_i is the weight of the asset S_i.

1.4 Option Prices as Expected Value of Payoff

The price of a European options can be expressed as a function of its payoff, and in this case it can be expressed as the expected value of its discounted payoff, under some probability measure Q . Assuming that the risk-free interest rate r is continuously compounded then the price of the option is,

$$\text{price} = e^{-rT} E_Q(\text{payoff}), \tag{1.7}$$

where e^{-rT} is the discount factor. A univariate call option has payoff $\max(0, S(T) - K)$ and by expanding $S(T)$ to its expression (1.2) under the Black-Scholes model we have

$$\text{price} = e^{-rT} E_Q(max(0, \ S(0)e^{(r-\delta-\frac{1}{2}\sigma^2)T+\sigma W(T)} - K))$$

In the above equation we have the parameters $S(0), r, \delta, \sigma, K,$ and $W(T)$ is a normal random variable. After taking the expected value the only undetermined values are the parameters, which implies the European call option price can be expressed as a function C of the parameters $S(0), K, T, r, \delta, \sigma$. And we can use the Black-Scholes formula which is a deterministic formula to calculate the price of such options.

$$\text{price} = C\left(S(0), K, T, r, \delta, \sigma\right) = S(0)e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2). \tag{1.8}$$

Where N is the cumulative distribution function for standard normal distribution

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} dt$$

d_1, d_2 are the z-values and they are determined by the following formulas

$$d_1 = \frac{\ln(\frac{S(0)}{K}) + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad \text{and } d_2 = d_1 - \sigma\sqrt{T}.$$

Similarly the Black-Scholes formula for put option is defined as

$$P\left(S(0), K, T, r, \delta, \sigma\right) = Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1).$$

Chapter 2

Pricing of Bivariate Derivatives

In this Chapter we are going to introduce a new methodology for the approximated pricing of options under a bivariate Black-Scholes model. In particular we will apply this methodology to the pricing of spread options, for which there is no analytic expression.

In Section 3.1 we introduce the model for the underlying assets and derive some conditional distributions that we will use afterwards. In Section 3.2 we show that the price of a spread option can be obtained as the expected value of a known function g that depends on the spot price of only one of the two underlying assets. Because the computation of this expected value cannot be done analytically, in Section 3.3 we consider least squares polynomial approximation $\hat{g}$ of g . In section 3.4 we explain how to use the derived function $\hat{g}$ to obtain approximated values of the spread option price. In addition, we also report some numerical results under several sets of model parameters, including comparison with Monte Carlo estimations of the price.

2.1 Distribution of Conditional Random Variables

Let us consider a bivariate Black-Scholes model $S = (S_1, S_2)^T$ as defined in expression (1.5). Our objective is essentially to find the conditional distribution of $S_1(T)$ conditional on $S_2(T)$. First let us find the conditional distribution of $(W_1(T)|W_2(T))$. Suppose $\mathbf{W}(T) = (W_1(T), W_2(T))^T$, we know from (1.1) $\mathbf{W}(T) \sim N(0, T\Sigma)$ where

$$T\Sigma = \begin{bmatrix} T & \rho T \\ \rho T & T \end{bmatrix}.$$

Using known results (Refer to Appendix 5.4) we get

$$(W_1(T)|W_2(T)) \sim N\left(\rho W_2(T),\ (1-\rho^2)T\right).$$

In the expression (1.2) for $S_i(T)$, all parameters are given except for the normal random variable $W_i(T)$. Therefore, we can rearrange the expression such that $W_2(T)$ is a function of $S_2(T)$

$$W_2(T) = \frac{\ln\left(\frac{S_2(T)}{S_2(0)}\right) - \left(r - \frac{1}{2}\sigma_2^2\right)T}{\sigma_2}.$$

From this we want to find the distribution of $\left(\ln\left(S_1(T)\right) \middle| W_2(T)\right)$, suppose the mean of the conditional random variable is m and the variance is v^2

$$\left(\ln\left(S_1(T)\right) \middle| W_2(T)\right) \sim N\left(m, v^2\right),$$

where the variance v^2 is given by

$$\begin{aligned} v^2 &= Var\left(\ln\left(S_1(T)\right) \middle| W_2(T)\right) \\ &= Var\left(\ln\left(S_1(0)\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 W_1 \ \middle|\ W_2(T)\right) \\ &= Var\left(\sigma_1 W_1(T) \ \middle|\ W_2(T)\right) \\ &= \sigma_1^2\left(1-\rho^2\right)T \end{aligned} \tag{2.1}$$

and the mean m is given by

$$
\begin{aligned}
m &= E\left(\ln\left(S_1(T)\right) \middle| W_2\right) \\
&= E\left(\ln\left(S_1(0)\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 W_1 \ \middle| \ W_2\right) \\
&= \ln\left(S_1(0)\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 E\left(W_1 \ \middle| \ W_2\right) \\
&= \ln\left(S_1(0)\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 \rho W_2 \\
&= \ln\left(S_1(0)\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 \rho \left(\frac{\ln\left(\frac{S_2(T)}{S_2(0)}\right) - \left(r - \frac{1}{2}\sigma_2^2\right)T}{\sigma_2}\right) \\
&= \ln\left(S_1(0)\right) + rT - \frac{\sigma_1^2}{2}T + \frac{\sigma_1^2\rho^2}{2}T - \frac{\sigma_1^2\rho^2}{2}T + \sigma_1 \rho \left(\frac{\ln\left(\frac{S_2(T)}{S_2(0)}\right) - \left(r - \frac{1}{2}\sigma_2^2\right)T}{\sigma_2}\right) \\
&= \ln\left(S_1(0)\right) + rT - \frac{\sigma_1^2\left(1-\rho^2\right)}{2}T - \frac{\sigma_1^2\rho^2}{2}T + \sigma_1 \rho \left(\frac{\ln\left(\frac{S_2(T)}{S_2(0)}\right) - \left(r - \frac{1}{2}\sigma_2^2\right)T}{\sigma_2 T} - \frac{\sigma_1\rho}{2}\right)T \\
&= \ln\left(S_1(0)\right) + rT - \frac{\sigma_1^2\left(1-\rho^2\right)}{2}T - \sigma_1 \rho \left(\frac{\sigma_1\rho}{2} - \frac{\ln\left(\frac{S_2(T)}{S_2(0)}\right) - \left(r - \frac{1}{2}\sigma_2^2\right)T}{\sigma_2 T}\right)T \\
&= \ln\left(S_1(0)\right) + rT - \frac{\sigma_1^2\left(1-\rho^2\right)}{2}T - \sigma_1 \rho \left(\frac{\sigma_1\rho}{2} - \frac{W_2(T)}{T}\right)T \\
&= \ln\left(S_1(0)\right) + rT - \frac{\sigma'^2}{2}T - \delta' T \qquad (2.2)
\end{aligned}
$$

These expressions for the mean (2.2) and variance (2.1) for the bivariate case is essentially equivalent to the expressions in (1.4) with different parameters δ' and σ' given by

$$\sigma' = \sigma_1\sqrt{1-\rho^2}, \qquad (2.3)$$

$$\delta' = \sigma_1\rho\left(\frac{\sigma_1\rho}{2} - \frac{W_2(T)}{T}\right).$$

2.2 Pricing After Conditioning

Recall that the price of an option can be determined by the expected value of its payoff (1.7), and according to the law of iterated expectation we have

$$\begin{aligned}\text{price} &= E(e^{-rT}\text{payoff})\\ &= e^{-rT}E((S_1(T)-S_2(T)-K)^+)\\ &= e^{-rT}E(E(S_1(T)-S_2(T)-K)|S_2(T))\\ &= E(e^{-rT}E(S_1(T)-(S_2(T)+K))|S_2(T))\end{aligned} \tag{2.4}$$

Let us now focus on the term $e^{-rT}E(S_1(T)-(S_2(T)+K)|S_2(T))$

From the above expression (2.4) we can see that after conditioning this is equivalent to having a standard call option on the first asset, with a new strike price $S_2(T)+K$, so we can define a new parameter $K^{'}$ as the strike price after conditioning

$$K^{'} = K^{'}(S_2(T)) = S_2(T)+K. \tag{2.5}$$

From the previous section we know that the distribution of the conditional random variable $\left(\ln\left(S_1(T)\right)\middle|W_2(T)\right)$ is identical to the distribution of $\ln(S(T))$ with parameters δ' and σ', more specifically we can compare the expressions (2.2) and (2.1) to the distribution in (1.4). Notice that δ' can be written in terms of $S_2(T)$ as follows

$$\delta' = \delta'(S_2(T)) = \sigma_1\rho\left(\frac{\sigma_1\rho}{2} - \frac{\ln\left(\frac{S_2(T)}{S_2(0)}\right)-\left(r-\frac{1}{2}\sigma_2^2\right)T}{\sigma_2 T}\right) \tag{2.6}$$

Now we can conclude that $e^{-rT}E(S_1(T)-(S_2(T)+K)|S_2(T))$ is equivalent to the price of a standard call option with strike price K', volatility σ' and dividend yield δ'. Therefore,

$$e^{-rT}E((S_1(T)-S_2(T)-K)^+) = C\left(S_1(0), K'(S_2(T)), T, r, \delta'(S_2(T)), \sigma'\right)$$

Omitting the other parameters, define g as follows

$$g(S_2(T)) = C\left(S_1(0), K'(S_2(T)), T, r, \delta'(S_2(T)), \sigma'\right) \tag{2.7}$$

The spread option price can be now written as

$$\text{price} = E(g(S_2(T)))$$

2.3 Least Square Regression Approximation

The analytical expression for the function g is known, but it is complicated to calculate the spread option price $E(g(S_2(T)))$. We will use a polynomial approximation $\hat{g}$ of g such that the expected value can be easily computed, and we can approximate the price as follows

$$\text{price} = E(g(S_2(T))) \approx E(\hat{g}(S_2(T))) \tag{2.8}$$

We will use a least squares criteria to find the approximating polynomial $\hat{g}$ of g and to minimize the effect of numerical errors we center our polynomial around $S^* = E(S_2(T))$. In other words, for a fixed $m \in \mathbb{N}$ we will use a set of predetermined nodes $x_1, x_2, \ldots, x_N$ (with $N > m$) to find coefficients $\beta_0, \beta_1, \ldots, \beta_m$ such that the mean square error

$$\sum_{i=1}^{N} [g(x_i) - (\beta_0 + \beta_1(x_i - S^*) + \beta_2(x_i - S^*)^2 + \cdots + \beta_m(x_i - S^*)^m]^2$$

is minimized [7]. Finding the coefficients $\beta_0, \beta_1, \ldots, \beta_m$ is essentially a linear regression problem. We will construct the nodes $x_1, x_2, x_3, \cdots, x_n$ in a finite interval where the values of $S_2(T)$ will most likely be.

Our nodes are constructed as follows

$$x_i = S_2(0)e^{(r-\delta-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}z_i}.$$

where,

$$z_i = N^{-1}(p_i)$$
$$p_i = 0.0001 + \frac{0.9998}{n} i$$

Since most of the values under the standard normal distribution are clustered around the mean, the majority of $S_2(T)$ values should be clustered in some region. Our goal is to make sure the approximation is accurate within the clustered region of $S_2(T)$.

Let us consider the following parameters, risk-free interest rate $r = 0.03$, an asset S_2 with volatility $\sigma_2 = 0.4$, dividend yield $\delta_2 = 0$ and $S_2(0) = 80$ that expires in a quarter.

To verify the regions of constructed nodes, we first show a histogram of $S_2(T)$ using randomly generated z's and expression (1.2)

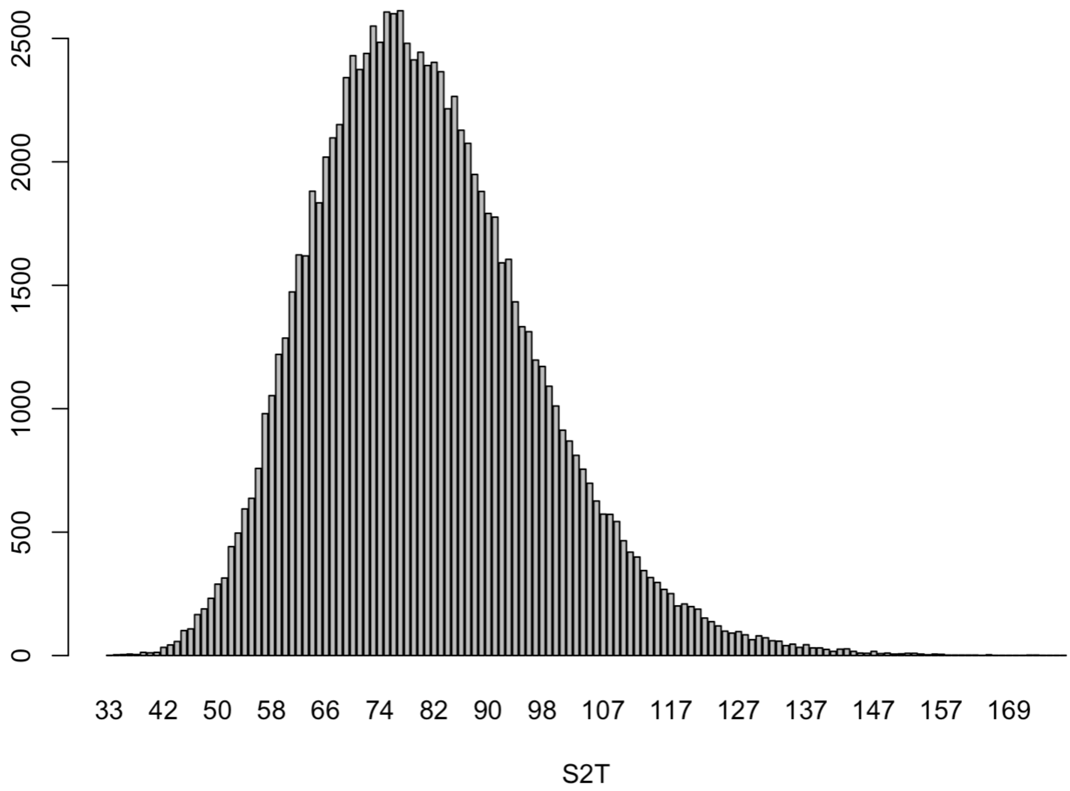


FIGURE 2.1: Randomly generated $S_2(T)$

Then we construct 1000 nodes using the method above, we have

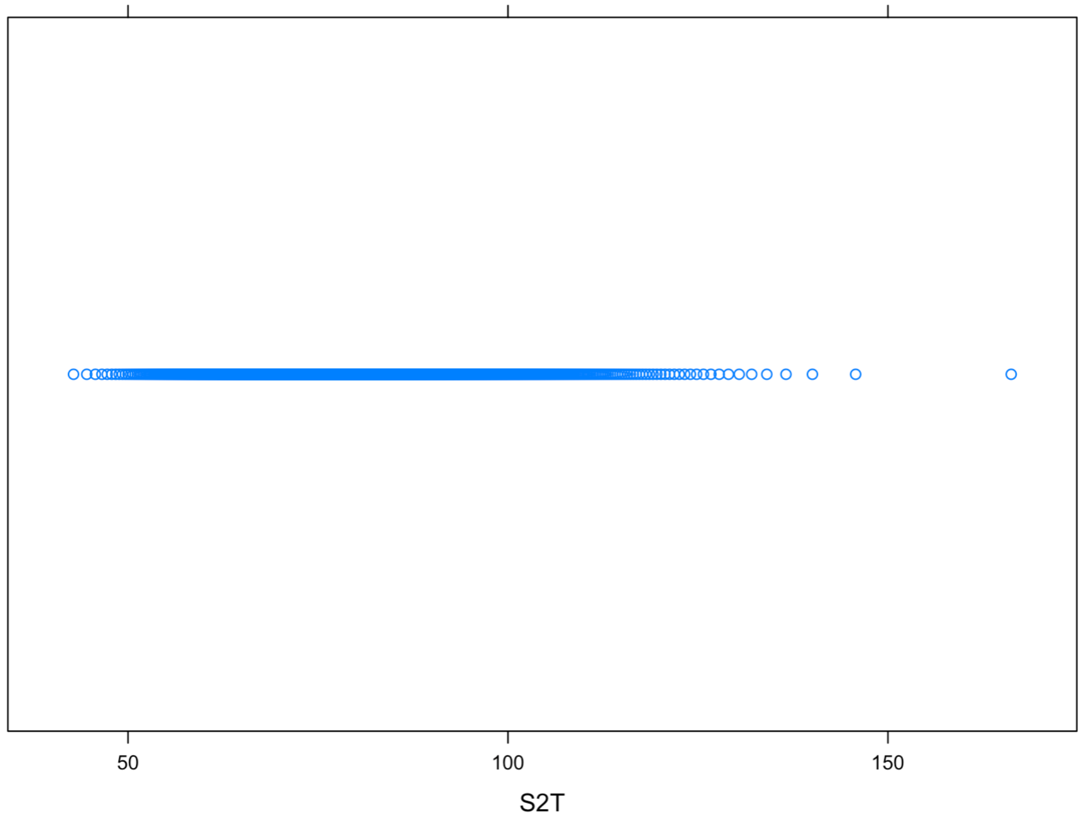


FIGURE 2.2: Constructed Nodes

We can see that the nodes in figure (2.2) we have most of the nodes in the interval $[50, 125]$ which correspond to the interval where most $S_2(T)$ values are in figure (2.1).

Polynomial approximation functions can have various degrees, while low degree could lead to under fitting the data points, a polynomial approximation with high degree could over fit the given points and become unreliable at other points. Another problem when computing high degree polynomial functions is that when dealing with extremely large or small numbers it may cause numerical problems for the computer. So we must carefully choose the correct degree.

Recall that g is a function of $S_2(T)$ which is essentially the conditional Black-Scholes formula (2.7), and $\hat{g}$ is a polynomial approximation of g.

$$\hat{g}(x) = \beta_0 + \beta_1(x - S^*) + \beta_2(x - S^*)^2 + \cdots + \beta_m(x - S^*)^m, \text{ for } m \in \mathbb{N} \quad (2.9)$$

The process of finding the optimal coefficients is achieved by using the “lm” function in R.

To compare the accuracy of the polynomial approximation function $\hat{g} = p_m$ of different degrees, we plotted the functions in comparison with g.

In a bivariate option, for S_1 we have volatility $\sigma_1 = 0.4$, dividend yield $\delta_2 = 0$ and $S_2(0) = 100$, the option expires in a quarter. Now we can plot the graphs for the function g and polynomial approximation functions of degree n, for $n = 2, 3, 4, 5, 6$.

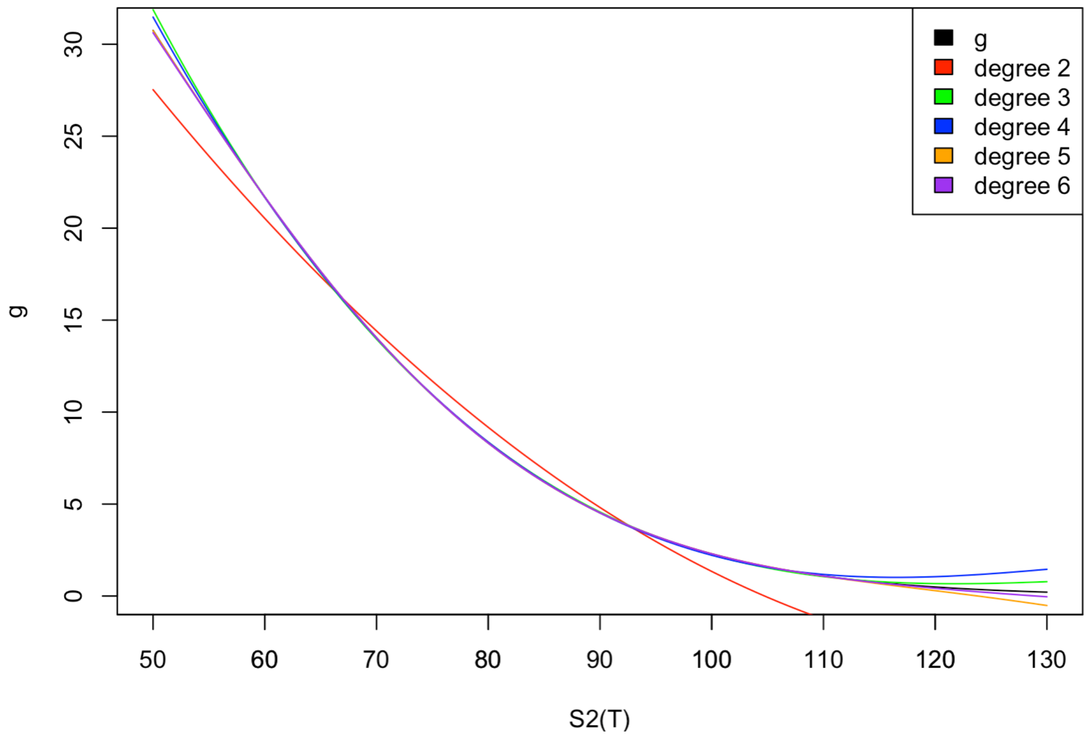


Figure 2.3: Graph of Approximation Functions

From the graph we can tell that in the desired interval [50,125] most of the approximation functions are reasonably accurate, especially in the center of the interval. Since the possibility of $S_2(T)$ to be outside the interval is minimal, even though the approximation functions are less accurate as the points extends outside the interval, the effect on the accuracy is small. Also notice that as we increase the degree of polynomials the accuracy also increases.

2.4 Approximated Pricing

Recall that the price is approximated by the expected value of polynomial approximation functions (2.8) and the expression (2.9). To compute the price of bivariate options, we need to find $E(\hat{g}(S_2(T)))$

$$\begin{aligned}
E(\hat{g}(S_2(T))) &= E(\hat{\beta}_0 + \hat{\beta}_1(S_2(T) - S^*) + \hat{\beta}_2(S_2(T) - S^*)^2 + \cdots + \hat{\beta}_m(S_2(T) - S^*)^m) \\
&= E(\hat{\beta}_0) + E(\hat{\beta}_1(S_2(T) - S^*)) + E(\hat{\beta}_2(S_2(T) - S^*)^2) + \cdots \\
&\quad + E(\hat{\beta}_m(S_2(T) - S^*)^m) \\
&= \hat{\beta}_0 + \hat{\beta}_1 E(S_2(T) - S^*) + \hat{\beta}_2 E((S_2(T) - S^*)^2) + \cdots + \hat{\beta}_m E((S_2(T) - S^*)^m)
\end{aligned}$$

Therefore,

$$\begin{aligned}
E((S_2(T) - S^*)^n) &= E\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}S_2(T)^{n-i}(S^*)^i\right) \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}E(S_2(T)^{n-i})(S^*)^i
\end{aligned}$$

Therefore, we can express the mth degree approximated price of a bivariate option as

$$\text{price} = \sum_{k=0}^{m}\hat{\beta}_k\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}E(S_2(T)^{k-i})(S^*)^i\right) \tag{2.10}$$

where $E(S_2(T)^a)$ and $a = k - i$ can be expressed as

$$E(S_2(T)^a) = S_2(0)^a e^{a(r-\frac{1}{2}\sigma)T + \frac{a^2}{2}\sigma^2 T}. \tag{2.11}$$

For a given set of parameters, to check the accuracy of the approximated prices, we compare them to the pricing results of two Monte Carlo approaches, the basic Monte Carlo method and the conditional Monte Carlo method.

For basic Monte Carlo method, we use the fact that

$$\text{price} = e^{-rT} E((S_1(0) - S_2(0) - K)^+)$$

So we determine the pricing results as follows:

1. generate n random normally distributed vectors $(z_{1i}, z_{2i})^T$, for $i = 1, 2, \cdots, n$ with mean 0 and co-variance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.
2. $S_{ki}(T) = S_k(0)e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma\sqrt{T}z_{ki}}$, where $k = 1, 2$
3. price $= \frac{\sum_{i=1}^{n} e^{-rT}(S_{1i}(T)-S_{2i}(T)-K)^+}{n}$

For the conditional Monte Carlo method, we use the fact from expression (2.4). And the process of conditional Monte Carlo method is shown as follows:

1. randomly generate n normally distributed random numbers z_{2i} for $i = 1, 2, \cdots, n$
2. $S_{2i}(T) = S_2(0)e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma\sqrt{T}z_{2i}}$
3. price $= \frac{\sum_{i=0}^{n} C\left(S_1(0), K_i', T, r, \delta_i'(S_2(T)), \sigma'\right)}{n} = \frac{\sum_{i=0}^{n} g(S_{2i}(T))}{n}$

For a set of parameters in we calculate the expected value of the polynomial approximation functions of degree 5 and 6, we get the following results in comparison with Monte Carlo methods results.

For the set of parameters:

$$\rho = 0,\ K = 20,\ t = 0.25,\ r = 0.03$$

$$S_1(0) = 100,\ \delta_1 = 0,\ S_2(0) = 80,\ \delta_2 = 0$$

	Basic M.C. [95% C.I.]	**Cond. M.C. [95% C.I.]**	**5th Degree [Std. Err.]**	**6th Degree [Std. Err.]**
$\sigma_1 = 0.4$	10.302	10.270	10.257	10.257
$\sigma_2 = 0.4$	[10.131,10.472]	[10.190,10.350]	[0.030]	[0.008]
$\sigma_1 = 0.2$	5.113	5.183	5.176	5.177
$\sigma_2 = 0.2$	[5.031,5.196]	[5.140,5.226]	[0.019]	[0.015]
$\sigma_1 = 0.2$	7.548	7.653	7.567	7.568
$\sigma_2 = 0.4$	[7.435,7.661]	[7.567,7.739]	[0.066]	[0.055]
$\sigma_1 = 0.4$	8.637	8.699	8.647	8.647
$\sigma_2 = 0.2$	[8.597,8.677]	[8.549,8.850]	[0.001]	[0.001]

TABLE 2.1: Regression nodes: 128; Monte Carlo Paths: 32768

For the set of parameters:

$$\rho = 0,\ K = 10,\ \sigma_1 = 0.2,\ \sigma_2 = 0.2,\ r = 0.03$$

$$S_1(0) = 110,\ \delta_1 = 0,\ S_2(0) = 110,\ \delta_2 = 0$$

	Basic M.C. [95% C.I.]	**Cond. M.C. [95% C.I.]**	**5th Degree [Std.Err.]**	**6th Degree [Std.Err.]**
t=0.25	2.420	2.488	2.469	2.468
	[2.358,2.483]	[2.455,2.521]	[0.090]	[0.039]
t=0.5	4.657	4.706	4.714	4.713
	[4.554,4.761]	[4.652,4.760]	[0.064]	[0.010]
t=0.75	6.587	6.590	5.216	5.218
	[5.162,5.246]	[5.187,5.218]	[0.068]	[0.011]
t=1	8.292	8.135	8.136	8.134
	[8.052,8.219]	[8.126,8.458]	[0.168]	[0.040]

TABLE 2.2: Regression nodes: 256; Monte Carlo Paths: 32768

Notice that for different set of parameters the accuracy for polynomial approximations are different. The different values of σ_1 has little impact, however the accuracy decrease as the value of σ_2 increases, due to the wider range of future $S_2(T)$ values caused by increasing volatility. Similarly the increasing time to expiration also tends to decrease the accuracy of the approximations, since it provides a wider range for future $S_2(T)$ values. Overall the results are sufficiently accurate, especially for the options that contains less volatile assets.

Chapter 3

Pricing of Trivariate Derivatives

In this Chapter we are going to expend the methodology from the bivariate case and apply it to the pricing of three-asset spread options which does not have an analytic expression.

In section 4.1 we cover the model for the trivaraite spread option and included the derivation of the conditional random variable. In section 4.2 we introduce the payoff of three-asset spread option and its price as the expected value of payoff, then we derived the modified parameters by comparing the distribution of the conditional normal distribution from section 4.1. In section 4.3 we approximate the modified Black-Scholes formula using least square regression method and compared the approximation function to the modified Black-Scholes formula. In section 4.4 we show the derivation of the expected value of the approximating function and compare pricing results with two Monte Carlo methods.

3.1 Distribution of Trivariate Conditional Random Variable

Let us now consider a trivariate Black-Scholes model $S = (S_1, S_2, S_3)^T$ as defined in expression (1.5). In the trivariate case we are looking for the conditional distribution of $S_1(T)$ conditional on $S_2(T), S_3(T)$. We want to find the distribution of conditional random variable $(W_1(T)|W_2(T), W_3(T))$. Suppose $\mathbf{W}(T) = (W_1(T), W_2(T), W_3(T))^T$, from (1.1) we have $\mathbf{W}(T) \sim N(0, T\Sigma)$ where

$$T\Sigma = \begin{bmatrix} T & \rho_{1,2}T & \rho_{1,3}T \\ \rho_{1,2}T & T & \rho_{2,3}T \\ \rho_{1,3}T & \rho_{2,3}T & T \end{bmatrix}$$

According to section 5.4 in Appendix we have that the distribution of the trivariate conditional normal random variable is $(W_1(T)|W_2(T), W_3(T)) \sim N(\bar{\mu}, \bar{\Sigma})$.

$$\begin{aligned}
\bar{\mu} &= 0 + \begin{bmatrix} \rho_{1,2}T & \rho_{1,3}T \end{bmatrix} \begin{bmatrix} \frac{1}{T(1-\rho_{2,3}^2)} & \frac{-\rho_{2,3}}{T(1-\rho_{2,3}^2)} \\ \frac{-\rho_{2,3}}{T(1-\rho_{2,3}^2)} & \frac{1}{T(1-\rho_{2,3}^2)} \end{bmatrix} \begin{bmatrix} W_2(T) \\ W_3(T) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_{1,2}-\rho_{1,3}\rho_{2,3}}{1-\rho_{2,3}^2} & \frac{\rho_{1,2}-\rho_{1,3}\rho_{2,3}}{1-\rho_{2,3}^2} \end{bmatrix} \begin{bmatrix} W_2(T) \\ W_3(T) \end{bmatrix} \\
&= \frac{W_2(T)\left(\rho_{1,2} - \rho 1, 3\rho_{2,3}\right) + W_3(T)\left(\rho_{1,3} - \rho 1, 2\rho_{2,3}\right)}{1 - \rho_{2,3}^2}
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\bar{\Sigma} &= T - \begin{bmatrix} \frac{\rho_{1,2}-\rho_{1,3}\rho_{2,3}}{1-\rho_{2,3}^2} & \frac{\rho_{1,3}-\rho_{1,2}\rho_{2,3}}{1-\rho_{2,3}^2} \end{bmatrix} \begin{bmatrix} \rho_{1,2}T \\ \rho_{1,3}T \end{bmatrix} \\
&= T - T\left(\frac{\rho_{1,2}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3}}{1 - \rho_{2,3}^2} + \frac{\rho_{1,3}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3}}{1 - \rho_{2,3}^2}\right) \\
&= \left(1 - \frac{\rho_{1,2}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3} + \rho_{1,3}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3}}{1 - \rho_{2,3}^2}\right) T
\end{aligned} \tag{3.2}$$

From this we want to find the distribution of the conditional normal random variable $(\ln(S_1(T)|W_2(T), W_3(T)) \sim N(m, v^2)$.

$$\begin{aligned}
v^2 &= Var(\ln(S_1(T))|W_2(T), W_3(T)) \\
&= Var(\sigma_1 W_1(T)|W_2(T), W_3(T)) \\
&= \sigma_1^2 \left(1 - \frac{\rho_{1,2}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3} + \rho_{1,3}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3}}{1 - \rho_{2,3}^2}\right) T
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
m &= E\left(ln(S_1(T)|W_2(T), W_3(T))\right) \\
&= E\left(\ln\left(S_{01}\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 W_1(T) \ \middle|\ W_2(T),\ W_3(T)\right) \\
&= ln(S_{01}) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 \frac{A}{1-\rho_{2,3}^2} \\
&= ln(S_{01}) + rT - \frac{1}{2}\sigma_1^2\left(1 - \frac{B}{1-\rho_{2,3}^2}\right)T\frac{\sigma_1 T}{1-\rho_{2,3}^2}\left(\frac{\sigma_1}{2}\left(B\right) - \frac{A}{T}\right)
\end{aligned} \tag{3.4}
$$

$$
A = \rho_{1,2}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3} + \rho_{1,3}^2 - \rho_{1,2}\rho_{1,3}\rho_{2,3} \tag{3.5}
$$

$$
B = W_2(T)(\rho_{1,2} - \rho_{1,3}\rho_{2,3}) + W_3(T)(\rho_{1,3} - \rho_{1,2}\rho_{2,3}) \tag{3.6}
$$

These expressions for the mean and variance of the conditional random variable $\ln(S_1(T)|W_2(T), W_3(T)$ is essentially equivalent to the expressions in (1.4) where δ and σ is replaced by the parameters δ' and σ' where

$$
\sigma' = \sigma_1\sqrt{1 - \frac{A}{1-\rho_{2,3}^2}}
$$

$$
\delta' = \frac{\sigma_1}{1-\rho_{2,3}^2}\left(\frac{\sigma_1}{2}A - \frac{B}{T}\right)
$$

3.2 Pricing After Conditioning

From expression (1.7), we get the price of trivariate case as the expected value of payoff. And for a three-asset spread option we have the payoff as $(S_1(T) - S_2(T) - S_3(T) - K)^+$

$$
\begin{aligned}
\text{price} &= E(e^{-rT}\text{payoff}) \\
&= E(e^{-rT}E(S_1(T) - S_2(T) - S_3(T) - K)|S_2(T), S_3(T))
\end{aligned}
$$

We now focus on the term $e^{-rT}E(S_1(T) - S_2(T) - S_3(T) - K)|S_2(T), S_3(T)$, we get that after conditioning this is equivalent to having a standard call option on the first asset with a new strike price $S_2(T) + S_3(T) + K$, denoted K' as the strike price after conditioning

$$K' = K'(S_2(T), S_3(T)) = S_2(T) + S_3(T) + K \tag{3.7}$$

Notice that the distribution of the conditional random variable $\ln(S_1(T)|W_2(T), W_3(T)$ is identical to the distribution of $\ln(S(T))$ with parameters δ' and σ', and that the parameter δ' can be written in terms of $S_2(T), S_3(T)$ as follows

$$\delta' = \delta'(S_2(T), S_3(T)) = \frac{\sigma_1}{1 - \rho_{2,3}^2}\left(\frac{\sigma_1}{2}A - \frac{B}{T}\right)$$

With A (3.5) and B (3.6). Then we can conclude that $e^{-rT}E(S_1(T) - S_2(T) - S_3(T) - K)|S_2(T), S_3(T)$ is equivalent to the price of a standard call option with strike price K', volatility σ' and dividend yield δ'. Therefore,

$$\begin{aligned} &e^{-rT}E(S_1(T) - S_2(T) - S_3(T) - K)|S_2(T), S_3(T)) \\ &= C\left(S_1(0), K'(S_2(T), S_3(T)), T, r, \delta'(S_2(T), S_3(T)), \sigma'\right) \end{aligned}$$

And we can define g as follows

$$g(S_2(T), S_3(T)) = C(S_1(0), K'(S_2(T), S_3(T)), T, r, \delta(S_2(T), S_3(T)), \sigma')$$

And the three-asset spread option price can be expressed as

$$\text{price} = E(g(S_2(T), S_3(T)))$$

3.3 Least Square Regression Approximation

Similarly in the trivariate case we want to find the expected value $E(g(S_2(T), S_3(T)))$ using the approximation function $\hat{g}$ of g, such that the price of trivariate option can easily be approximated

$$\text{price} = E(g(S_2(T), S_3(T))) \approx E(\hat{g}(S_2(T), S_3(T)))$$

Similar to the bivariate case, we will use the least square criteria to find the approximating bivariate polynomial function $\hat{g}$. Notice in the above graph when the sum $S_2(T) + S_3(T)$ remains unchanged the price of the trivariate option is barely affected by the different values of each $S_2(T)$ and $S_3(T)$. So we can use the set of predetermined nodes $x_i + y_i$ combined with x_i, y_i (for $i = 1, 2, \cdots, N$ and $N > m$) to find the coefficients $\beta_0, \beta_1, \cdots, \beta_m$ such that

$$\sum_{i=1}^{N} [g(x_i, y_i) - (\beta_0 p_0(x_i + y_i) + \beta_1 p_1(x_i + y_i) + \beta_2 p_2(x_i + y_i) + \cdots + \beta_m p_m(x_i + y_i))]^2$$

is minimized, where $p_j(x_i, y_i)$ are polynomials.

Finding the approximationg function $\hat{g}$ for the trivariate case is also a linear regression problem that considers the nodes $x_i + y_i$, x_i and y_i. The nodes should be clustered around a region of values for $S_2(T)$ and $S_3(T)$.

To generate the the nodes x_i and y_i, we first divide the interval $[0.005, 0.995]$ in m partitions and follow the procedure:

$$p_{2i} = 0.005 + \frac{0.995 - 0.005}{m} i, \text{ for } i = 1, 2, \cdots, m$$

$$p_{3i} = 0.005 + \frac{0.995 - 0.005}{m} i$$

then compute the z-values

$$z_{2i} = N^{-1}(p_{2i})$$

$$z_{3i} = z_{2i}\rho_{2,3} + \sqrt{1-\rho_{2,3}}N^{-1}(p_{3i}), \ \ \rho_{2,3} \text{ is the correlation coefficient}$$

And to further squeeze the values of z_2 and z_3 such that the $S_2(T)$ and $S_3(T)$ values produced are more clustered in the region where most of their values will be, we adjusted the values using $\theta = \frac{\sin^{-1}\rho_{2,3}}{2}$.

$$z'_{2i} = z_{2i}\cos\theta + z_{3i}\sin\theta, \ \text{and} \ z'_{3i} = z_{2i}\sin\theta + z_{3i}\cos\theta$$

Then the nodes are calculated using expression (1.2) with the adjusted z'_{2i} and z'_{3i} values.

By randomly generate two sets of standard normal random variables z_2 and z_3, after applying the stock price expression (1.2) we can get the plot of randomly generated $S_2(T)$ and $S_3(T)$ values

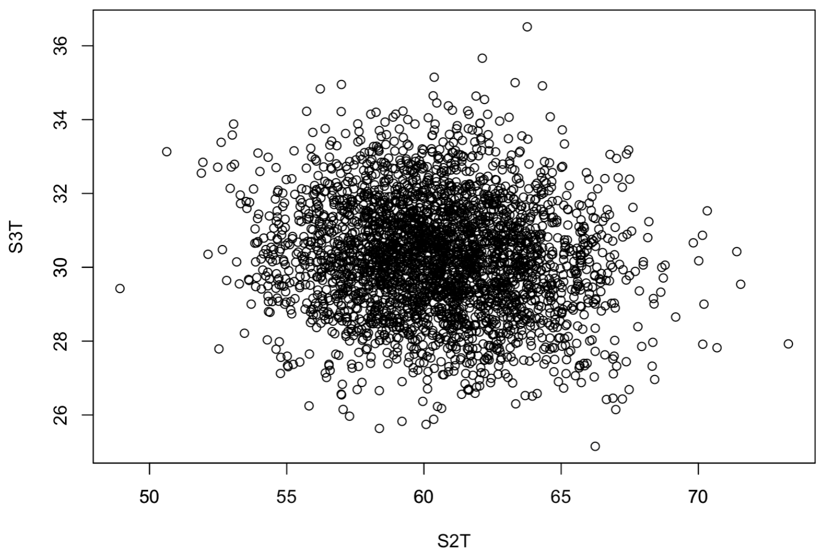


FIGURE 3.1: Generated Nodes

Then using the above method we can generate a plot of the nodes, and compare its region to the randomly generated $S_2(T)$ and $S_3(T)$ values.

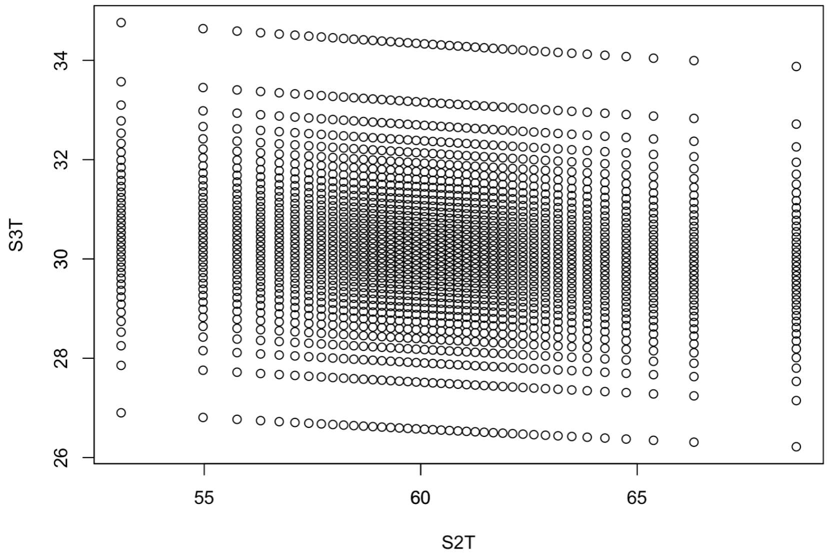


FIGURE 3.2: Calculated Nodes

Notice in graph (3.2) the nodes are concentrated in the same region as the generated pairs $(S_2(T), S_3(T))$ in graph (3.1), more specifically for $S_2(T) \in [40, 80]$ and $S_3(T) \in [30, 80]$. Essentially, by selecting those nodes we are making sure that $\hat{g}$ and g are close in this region

Notice in the graph below when the sum $S_2(T) + S_3(T)$ remains unchanged the price of the trivariate option is barely affected by the different values of each $S_2(T)$ and $S_3(T)$. So we can use the set of predetermined nodes $x_i + y_i$ combined with x_i, (for $i = 1, 2, \cdots, N$ and $N > m$) to find the coefficients $\beta_0, \beta_1, \cdots, \beta_m$. we have tested with different form of the nodes and find that among the nodes and degrees we have tested the following approximating function yields the best result.

$$\hat{g}(x, y) = \beta_0 + \beta_1(x + y) + \cdots + \beta_5(x + y)^5 + \beta_6 x + \beta_7 x^2 \tag{3.8}$$

where $x = S_2(T)$ and $y = S_3(T)$.

While the graphs of the approximating function $\hat{g}$ compared to the modified Black-Scholes formula g can provide some idea on the accuracy.

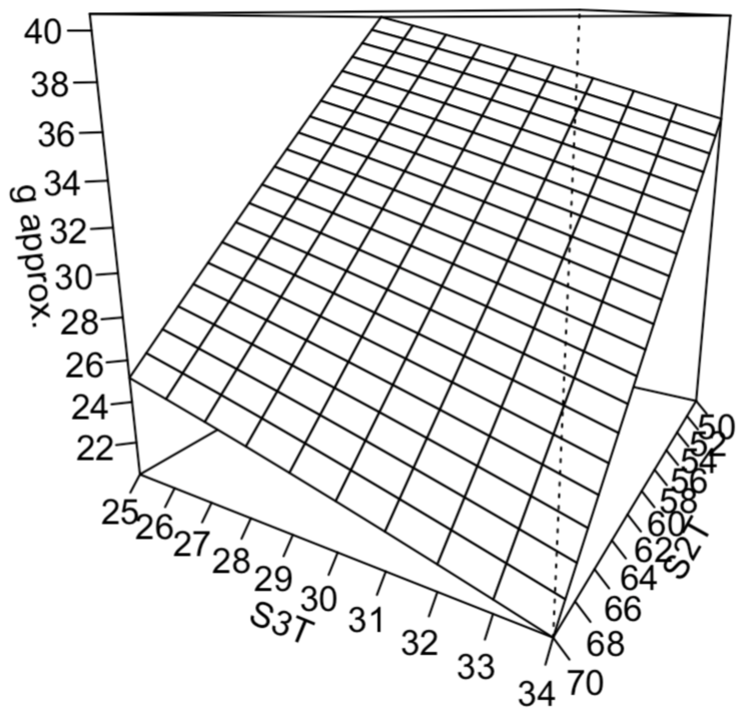


Figure 3.3: Modified Black-Scholes

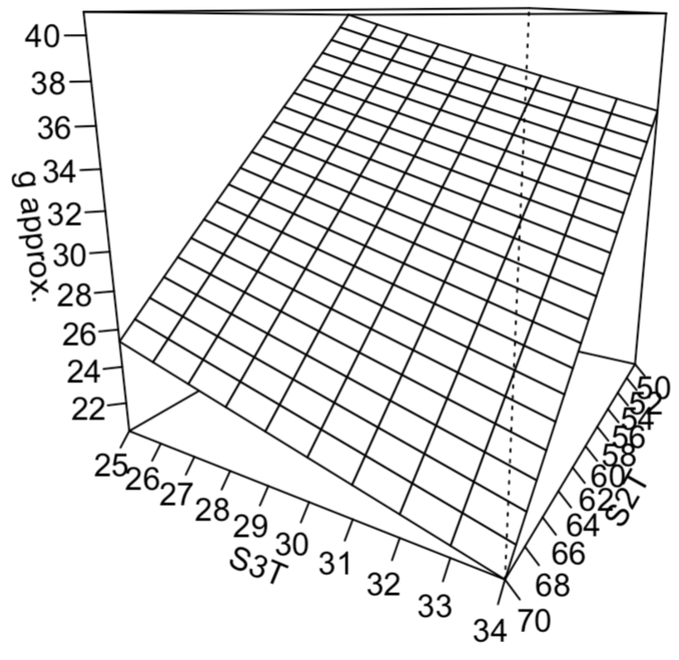


Figure 3.4: Approximation Function

To better illustrate the accuracy in one graph, the plot in trivariate case isfocused on the error $g - \hat{g}$ instead of the functions.

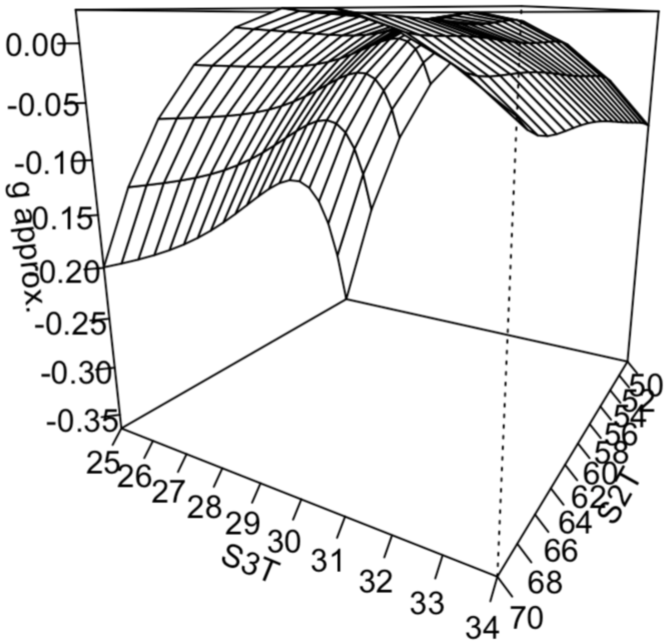


FIGURE 3.5: Error

From the graphs we can tell that the errors in the region are within the interval $[-0.35, 0]$, and even better at the center of the region. For the purpose of pricing the trivariate option, the approximating function is accurate within the desired region where most of the $S_2(T)$ and $S_3(T)$ values will be.

3.4 Approximated Price

Recall that the price of the option can be determined by the expected value of the discounted payoff (**??**). It can be approximated by the expected value of the approximating function $E(\hat{g})$.

$$E(\hat{g}(S_2(T), S_{(T))} = E(\sum_{i=0}^{5} \hat{\beta}_i (S_2(T) + S_3(T))^i + \sum_{j=5}^{6} \hat{\beta}_j x^{j-4})$$

$$= \sum_{i=0}^{5} \hat{\beta}_i E((S_2(T) + S_3(T))^i) + \sum_{j=5}^{6} \hat{\beta}_j E(x^{j-4})$$

where $x = S_2(T)$ and $y = S_3(T)$, therefore

$$\begin{aligned} E((S_2(T)+S_3(T))^n) &= E(\sum_{i=0}^{n}\binom{n}{i}S_2(T)^k S_3(T)^{n-i}) \\ &= \sum_{i=0}^{n}\binom{n}{i}E(S_2(T)^k S_3(T)^{n-i}) \end{aligned}$$

and the expression of $E(S_2(T)^j S_3(T)^k)$ is given by

$$S_2(0)^j S_3(0)^k e^{(j+k)rT - \frac{T}{2}\left(j\sigma_2^2(1-j)+k\sigma_3^2(1-k)\right)+jk\sigma_2\sigma_3\rho_{2,3}T}$$

The pricing result of trivariate option is also compared with the result of two Monte Carlo methods, basic Monte Carlo and conditional Monte Carlo. And we found that most results are reasonably accurate.

The values of $S_2(T)$ and $S_3(T)$ in conditional Monte Carlo in the trivariate case is generated in a similar way to the $S_1(T)$ and $S_2(T)$ values in the basic Monte Carlo method of bivariate case. Then taking mean of values obtained using the trivariate modified Black-Scholes formula to be the estimated price.

For the basic Monte Carlo, we need to first randomly choose a set of values for z_2 which following the standard normal distribution, then apply formula (**??**) to generate the set of z_3 values. And we can form two sets of values for $W_2(T)$ and $W_3(T)$, using the conditional normal distribution of $(W_1(T)|W_2(T), W_3(T))$ we get the mean (3.1) and variance (3.2). Then we can obtain a set of $W_1(T)$ values which are randomly selected. By apply the stock price formula (1.2) and the trivariate modified Black-Scholes formula, then the mean of the set of result is the estimated price by the conditional Monte Carlo method.

Now we compare the results from the $E(\hat{g})$ and the two Monte Carlo methods.

For the set of parameters:

$$K = 10,\ r = 0.03,\ \rho_{1,2} = 0.2,\ \rho_{1,3} = 0.2,\ \rho_{2,3} = 0.3,\ \delta = 0$$

$$S_1(0) = 100,\ \sigma_1 = 0.3,\ S_2(0) = 30,\ \sigma_2 = 0.4,\ S_3(0) = 40,\ \sigma_2 = 0.4$$

	Basic M.C. [95% C.I.]	**Cond. M.C.** [95% C.I.]	**Approx. Func.**
t=0.25	20.934 [20.892,20.976]	20.892 [20.874,20.911]	20.924
t=0.5	22.510 [22.454,22.566]	22.495 [22.473,22.518]	22.552
t=0.75	24.034 [23.968,24.100]	24.023 [23.998,24.048]	24.111
t=1	25.430 [25.353,25.503]	25.452 [25.426,25.479]	25.550

Table 3.1: Trivariate Pricing Results 1

For the set of parameters:

$$K = 10,\ r = 0.03,\ \rho_{1,2} = 0.2,\ \rho_{1,3} = 0.2,\ \rho_{2,3} = 0.3,\ t = 0.5,\ \delta = 0$$

$$S_1(0) = 100,\ \sigma_1 = 0.3,\ S_2(0) = 30,\ S_3(0) = 40$$

	Basic M.C. [95% C.I.]	**Cond. M.C.** [95% C.I.]	**Approx. Func.**
$\sigma_2 = 0.1$ $\sigma_3 = 0.1$	10.697 [10.674,10.719]	10.706 [10.697,10.715]	10.704
$\sigma_2 = 0.3$ $\sigma_3 = 0.3$	13.359 [13.324,13.394]	13.359 [13.329,13.388]	13.369
$\sigma_2 = 0.4$ $\sigma_3 = 0.2$	13.014 [12.981,13.048]	12.992 [12.965,13.020]	13.012
$\sigma_2 = 0.4$ $\sigma_3 = 0.3$	14.039 [14.002,14.077]	14.069 [14.036,14.101]	14.076
$\sigma_2 = 0.4$ $\sigma_3 = 0.3$	14.039 [14.002,14.077]	14.069 [14.036,14.101]	14.076
$\sigma_2 = 0.1$ $\sigma_3 = 0.4$	13.730 [13.695,13.766]	13.763 [13.733,13.793]	13.708
$\sigma_2 = 0.1$ $\sigma_3 = 0.3$	12.421 [12.390,12.452]	12.430 [12.406,12.454]	12.385

Table 3.2: Trivariate Pricing Results 2

From the tables (3.1) and (3.2) above, one can tell that accuracy depends on the parameters. As time to expiration increases the less accurate the pricing result will be, and when the volatility σ_3 of S_3 is large and difference of σ_2 and σ_3 is large the pricing result is not as accurate.

Chapter 4

Appendix

This chapter contains most of the critical preliminaries used throughout the thesis to make this thesis as self-contained as possible. This chapter is divided into 3 sections, containing preliminaries for Statistics, Finance and Polynomial Approximation.

4.1 Normal Random Variable

In this thesis, we mainly considered the random variables in normal distribution, a distribution with bell shape and with most of the variables concentrated around the mean.

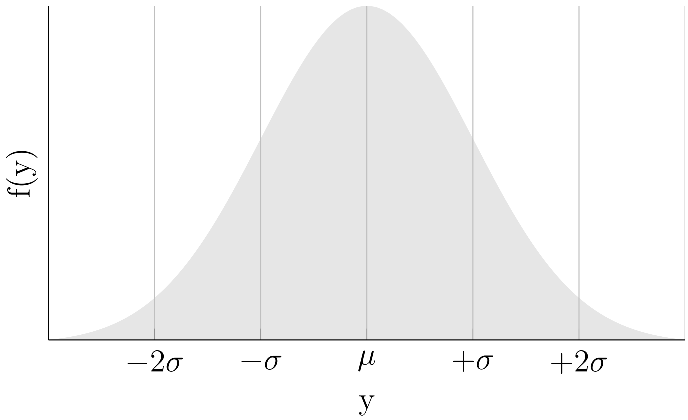
A random variable Y is said to have a normal probability distribution if and only if, for $\sigma > 0$ and $-\infty < \mu < \infty$, the density function of Y is:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}, \qquad -\infty < y < \infty.$$

If Y is a normally distributed random variable with parameters μ and σ, denoted $Y \sim N(\mu, \sigma)$, then

$$E(Y) = \mu \quad \text{and} \quad V(Y) = \sigma^2.$$

Where $E(Y)$ and $V(Y)$ is the expected value of and variance of normal random variable Y.



4.2 Log-Normal Distribution

In statistics, the log-normal distribution is a probability distribution of a random variable which have a normally distributed natural logarithm. That is if X follows the log-normal distribution, then the random variable $ln(X)$ is normally distributed. Reversely, if Y is normally distributed, then the random variable e^Y follows a log-normal distribution.

For a log-normal random variable X, if $ln(X) \sim N(\mu, \sigma^2)$ then

$$E(X^t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}.$$

In the Black-Scholes model, for the price of a stock over time T we have a function $S(T)$ that is log-normally distributed

$$S(T) = S_0 e^{(r - \delta - \frac{1}{2}\sigma^2)T + \sigma W(T)}$$

Where $W(T) \sim N(0, T)$, therefore

$$ln(S(T)) = ln(S_0) + \left(rT - \delta T - \frac{1}{2}\sigma^2 T + \sigma W(T) \right)$$

is normally distributed.

4.3 Multivariate Normal Distribution

X is an n-dimensional multivariate normal random variable, X can be represented as a vector $X = (X_1, X_2, \cdots, X_n)^T$, where $X_1, X_2, \cdots, X_n$ are normal random variables, the multivariate probability function is given by

$$p(x_1, x_2, \cdots, x_n) = P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n).$$

And X follows the multivariate normal distribution, $X \sim N(\mu, \Sigma)$ where

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ . \\ \mu_n \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 & \cdots & \rho_{1,n}\sigma_1\sigma_n \\ \rho_{2,1}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2,n}\sigma_2\sigma_n \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \rho_{n,1}\sigma_n\sigma_1 & \rho_{n,2}\sigma_n\sigma_2 & \cdots & \sigma_n^2 \end{bmatrix}$$

4.4 Conditional Distributions

In statistics, given two jointly distributed random variables X and Y, the conditional probability distribution of X given Y, denoted $P(X|Y)$ is the probability distribution of X when Y is known.

In the general case we have $X_1, X_2, \cdots, X_n$, multiple jointly distributed random variables, the distribution of the conditional random variable $(X_1, \cdots, X_k | X_{k+1}, \cdots, X_n) \sim N(\bar{\mu}, \ \bar{\Sigma})$ where $\bar{\mu}, \ \bar{\Sigma}$ are given by X, μ, Σ

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ with sizes } \begin{bmatrix} k \times 1 \\ (n-k) \times 1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{ with sizes } \begin{bmatrix} k \times 1 \\ (n-k) \times 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \text{ with sizes } \begin{bmatrix} k \times 1 & 1 \times (n-k) \\ (n-k) \times 1 & (n-k) \times (n-k) \end{bmatrix}$$

Let a be the vector of values of $X_{k+1}, \cdots, X_n$, then $a = (x_{k+1}, \cdots x_n)^T$

$$\bar{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(a - \mu_2)$$

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

4.5 Least Square Approximation

Let $k \geq 1$ be an integer. Let the real valued functions $g, f_1, f_2, \ldots, f_n$ defined over some domain $D \subset \mathbb{R}^k$, be given. Consider also a finite set of points $S = \{x_1, x_2, \ldots, x_m\} \subset D$, where $m > n$. We would like to approximate the function g with linear combinations of the functions $f_1, f_2, \ldots, f_n$.

For coefficients $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^T \in \mathbb{R}^n$, consider the function $g_\beta = \sum_{i=1}^n \beta_i f_i$, and the square error over the points in S given by:

$$SE(\beta) = \sum_{j=1}^{m} (g(x_j) - g_\beta(x_j))^2$$

In this setting, the least squares approximation of the function g is $g_{\hat{\beta}}$ where $\hat{\beta}$ minimizes $SE(\beta)$.

This is a particular case of well known approximation problems that have been solved in very general settings, and can be essentially reduced to a multiple regression problem.

If we define $y = (g(x_1), g(x_2), \ldots, g(x_m))^T$ and

$$X = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_m) & f_2(x_m) & \cdots & f_n(x_m) \end{bmatrix}$$

and we assume that the matrix X has full rank, the optimal coefficients $\hat{\beta}$ can be obtained as

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

4.6 Monte Carlo Method for Option Pricing

The Monte Carlo method is a computational algorithm that uses repeated random sampling to obtain numerical results. In the options pricing application it simulates possible future paths for underlying assets. It is modeled to follow geometric Brownian motion with a set of constant parameters. In this thesis, it is combined with the stock price function in the Black-Scholes model to compute the stock price at expiration of an option to obtain a pricing result. The final result is determined by taking the average of the collection of simulated stock prices.

The standard Monte Carlo method for option pricing can be divided into 3 steps:

- Randomly generate the future prices of the underlying assets.
- Calculate the payoff of the option for each of the generated underlying price scenarios.
- Discount the payoffs back to today and average them to determine the expected price.

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