A Piecewise Linear Spline Maximum Entropy Method for Frobenius-Perron Operators of Multi-dimensional Transformations

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Definitions

Let $(X, \mathcal{B}, \mu)$ be a probability space and $\tau : X \to X$ be a measurable transformation.

Definition

Measure μ is said to be τ-**invariant** if and only if for every $E \in \mathcal{B}$,

$$\mu(\tau^{-1}E) = \mu(E).$$

Alternatively, we may say that τ is a **measure-preserving** transformation.

Definition

Measure-preserving transformation τ is **ergodic** if for every $E \in \mathcal{B}$ with $\tau^{-1}E = E$, either $\mu(E) = 0$ or $\mu(E) = 1$.

Theorem (Birkhoff's Ergodic Theorem)

Let $(X, \mathcal{B}, \mu)$ be a measure space, τ an ergodic transformation and suppose g is a μ-integrable function, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\tau^i(x)) = \frac{1}{\mu(X)} \int_X g \, d\mu \quad a.e.$$

Corollary

Let $(X, \mathcal{B}, \mu)$ be a probability space and τ an ergodic transformation, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(\tau^i(x)) = \mu(E) \quad \mu - a.e.$$

Simple Application

Baker's Map

Consider the (folded) baker's map given by

$$\tau(x, y) = \begin{cases} (2x, y/2), & 0 \leq x \leq 1/2 \\ (2 - 2x, 1 - y/2), & 1/2 < x \leq 1 \end{cases}$$

This transformation preserves two-dimensional lebesgue measure.

$$\overset{\tau}{\Longrightarrow}$$

Tools
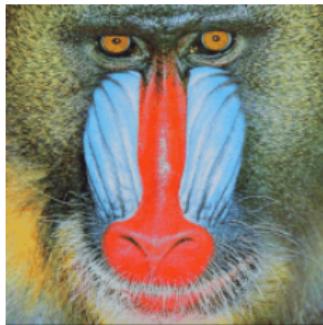
Density Functions

If μ is absolutely continuous with respect to Lebesgue measure λ (the usual notion of length/area/volume), then we can write

$$\mu(E) = \int_E f \, d\lambda, \quad \text{for some function } f.$$

We refer to f as the *density of* μ.

Definition

The Frobenius-Perron operator P_τ is defined by

$$\int_E P_\tau f \, d\lambda = \int_{\tau^{-1}E} f \, d\lambda.$$

If $P_\tau f = f$, then the measure μ defined by f is invariant.

Methods

Histogram

Direct application of Birkhoff's Ergodic theorem. Partition the interval into "bins" E_i. For each bin, calculate

$$\frac{1}{N} \sum_{i=0}^{N-1} \chi_{E_i}(\tau^i(x)) = \mu(E_i),$$

for large integer N.

Ulam's method

Uses a finite approximation of the Frobenius-Perron operator. Divide interval into n equal subintervals $I_1, I_2, \ldots, I_n$ and define

$$P_{ij} = \lambda(I_i \cap \tau^{-1}I_j)/\lambda(I_i),$$

and

$$P_n \chi_i = \sum_{j=1}^{n} P_{ij}\chi_j.$$

Find fixed point f_n of P_n

B-splines

Nodes

Begin by generating the following $n + 1$ nodes

$$a_{-1} < a_0 < \ldots < a_{n-2} < a_{n-1}, \quad \text{where } a_i = i/n.$$

2D Linear B-splines

Define

$$L(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \\ 0, & x \notin [0, 2] \end{cases}$$

Generate the following $(n + 1)^2$ linear B-splines

$$B_{i,j}(x, y) = L\left(\frac{x - a_i}{1/n}\right) L\left(\frac{y - a_j}{1/n}\right), \quad i, j \in \{-1, 0, \ldots, n - 1\}$$
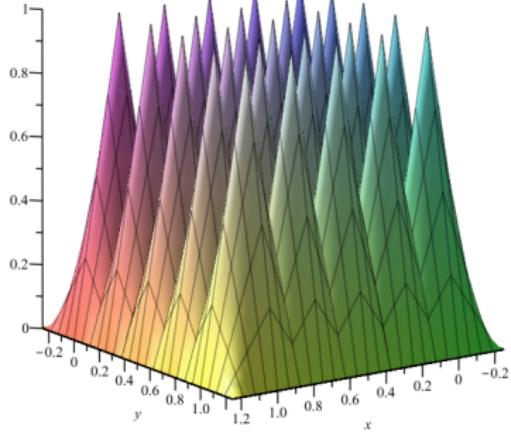
(a) B-splines for $n = 4$
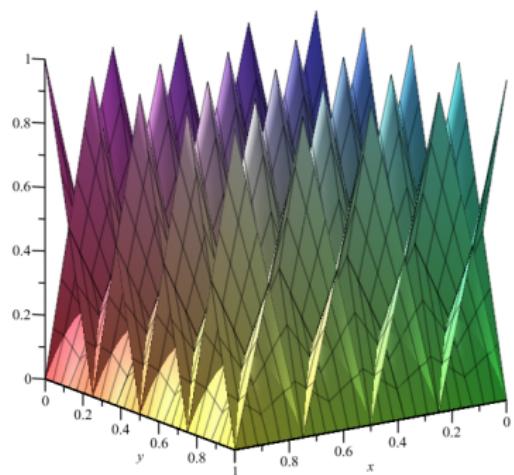
(b) B-splines restricted to $[0, 1]^2$

Maximum Entropy

Definition (Boltzmann Entropy)

The **Boltzmann entropy** H of a nonnegative integrable function f is defined to be

$$H(f) = - \int_{[0,1]^k} f(\vec{x}) \ln f(\vec{x}) \, d\vec{x}$$

Our constrained maximization problem is of the following form

$$\max \left\{ H(f) : \|f\| = 1, \int_0^1 \int_0^1 f(x,y) B_i(x,y) \, dxdy = m_i \right\}$$

To which our solution is of the form

$$f_n(x,y) = e^{\sum_{i=1}^{(n+1)^2} c_i B_i(x,y)}$$

Algorithm

1. Choose n and partition $[0,1]^2$ into n^2 sets
2. Approximate moments m_i using

$$m_i \approx \frac{1}{N} \sum_{j=0}^{N-1} B_i(\tau^j(x,y)), \quad i = 1, 2, \ldots, (n+1)^2$$

   for some large N and some $x, y \in [0,1]$
3. Solve the following system for $c_1, c_2, \ldots, c_{(n+1)^2}$

$$\int_0^1 \int_0^1 B_i(x,y) e^{\sum_{j=1}^{(n+1)^2} c_j B_j(x,y)} \, dxdy = m_i, \quad i = 1, 2, \ldots, (n+1)^2$$

4. Form density $f_n(x,y) = e^{\sum_{i=1}^{(n+1)^2} c_i B_i(x,y)}$

Example

Problem

Consider the transformation given by $\tau(x, y) = (S(x), T(y))$, where

$$S(x) = \left(\frac{1}{8} - 2\left|x - \frac{1}{2}\right|^3\right)^{1/3} + 1/2,$$

$$T(y) = \begin{cases} \frac{2y}{1-y^2}, & 0 \leq y < \sqrt{2} - 1 \\ \frac{1-y^2}{2y}, & \sqrt{2} - 1 \leq y \leq 1 \end{cases}$$

The exact density is given by

$$f^*(x, y) = \frac{48(x - 1/2)^2}{\pi(1 + y^2)}.$$

Maximum Entropy Errors

| n	L^1-norm error
2	1.4×10^{-1}
4	3.9×10^{-2}
8	1.1×10^{-2}
16	2.9×10^{-3}
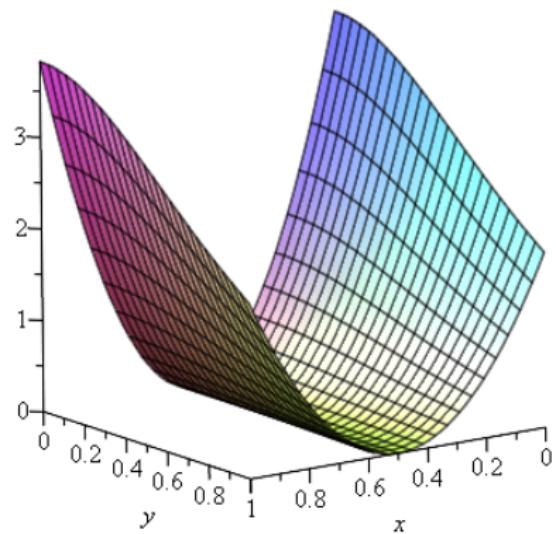
Figure: f_{16} and exact density f^*

Concluding Remarks

The linear spline maximum entropy method described for multidimensional transformations is easy to implement but can suffer from a few problems. Namely, the use of Birkhoff's ergodic theorem means there is the possibility of choosing an initial point that does not obey the desired relationship between time average and space average. There is also the concern of how the approximation of the moments will affect the approximation of our density. Much of the computational cost of this algorithm comes from the solution of the system using some system solver such as Newton's method.

Special Thanks